SOLUÇÃO PROBLEMA 2.86 - RAO 4ª Edição Pearson Brasil

2.86 Admitindo que o ângulo de fase seja zero, mostre que a resposta $x(t)$ de um sistema subamortecido com um grau de liberdade alcança um valor máximo quando

$$\text{sen } \omega_d t = \sqrt{1-\zeta^2}$$

e um valor mínimo quando

$$\text{sen } \omega_d t = -\sqrt{1-\zeta^2}$$

Mostre também que as equações das curvas que passam pelos valores máximos e mínimos de $x(t)$ são dadas, respectivamente, por

$$x = \sqrt{1-\zeta^2} X e^{-\zeta\omega_n t}$$

e

$$x = -\sqrt{1-\zeta^2} X e^{-\zeta\omega_n t}$$

Problemas com solução da vibração livre subamortecida com $x(0)=0$:

$$x(t) = X\, e^{-\zeta\omega_n t} \sin \omega_d t \quad \text{where} \quad \omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

For maximum or minimum of $x(t)$,

$$\frac{dx}{dt} = X\, e^{-\zeta\omega_n t}\left(-\zeta\omega_n \sin \omega_d t + \omega_d \cos \omega_d t\right) = 0$$

As $e^{-\zeta\omega_n t} \neq 0$ for finite t,

$$-\zeta\omega_n \sin \omega_d t + \omega_d \cos \omega_d t = 0$$

i.e. $\tan \omega_d t = \sqrt{1-\zeta^2}/\zeta$

Using the relation

$$\sin \omega_d t = \pm \frac{\tan \omega_d t}{\sqrt{1+\tan^2 \omega_d t}} = \pm \frac{\left(\sqrt{1-\zeta^2}/\zeta\right)}{\sqrt{1+\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)^2}} = \pm\sqrt{1-\zeta^2}$$

we obtain

$$\sin \omega_d t = \sqrt{1-\zeta^2}, \quad \cos \omega_d t = \zeta$$

and

$$\sin \omega_d t = -\sqrt{1-\zeta^2}, \quad \cos \omega_d t = -\zeta$$

$$\frac{d^2x}{dt^2} = X\, e^{-\zeta\omega_n t}\left[\zeta^2\omega_n^2 \sin \omega_d t - 2\zeta\omega_n\omega_d \cos \omega_d t - \omega_d^2 \sin \omega_d t\right]$$

when $\sin \omega_d t = \sqrt{1-\zeta^2}$ and $\cos \omega_d t = \zeta$,

$$\frac{d^2x}{dt^2} = - X e^{-\zeta \omega_n t} \omega_n^2 \sqrt{1-\zeta^2} < 0$$

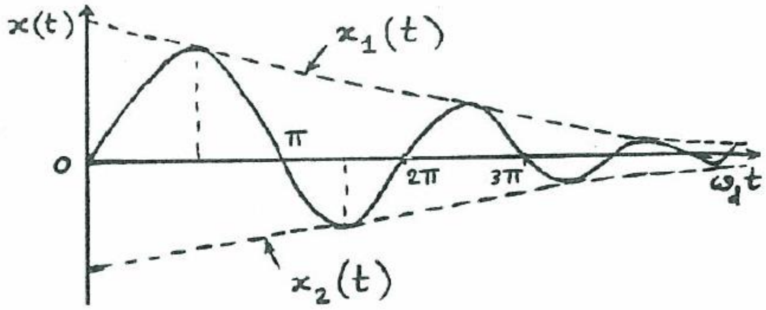
$\therefore$ $\sin \omega_d t = \sqrt{1-\zeta^2}$ corresponds to maximum of $x(t)$.

When $\sin \omega_d t = -\sqrt{1-\zeta^2}$ and $\cos \omega_d t = -\zeta$,

$$\frac{d^2x}{dt^2} = X e^{-\zeta \omega_n t} \omega_n^2 \sqrt{1-\zeta^2} > 0$$

$\therefore$ $\sin \omega_d t = -\sqrt{1-\zeta^2}$ corresponds to minimum of $x(t)$.

Enveloping curves:



Let the curve passing through the maximum (or minimum) points be

$$x(t) = C\, e^{-\zeta \omega_n t}$$

For maximum points, $t_{max} = \dfrac{\sin^{-1}\left(\sqrt{1-\zeta^2}\right)}{\omega_d}$

and

$$C\, e^{-\zeta \omega_n t_{max}} = X e^{-\zeta \omega_n t_{max}} \sin \omega_d t_{max}$$

i.e.

$$C = X \sqrt{1-\zeta^2}$$

$$\therefore\ x_1(t) = X \sqrt{1-\zeta^2}\, e^{-\zeta \omega_n t}$$

Similarly for minimum points, $t_{min} = \dfrac{\sin^{-1}\left(-\sqrt{1-\zeta^2}\right)}{\omega_d}$

and

$$C\, e^{-\zeta \omega_n t_{min}} = X e^{-\zeta \omega_n t_{min}} \sin \omega_d t_{min}$$

i.e. $C = -X\sqrt{1-\zeta^2}$

$$\therefore\ x_2(t) = -X\sqrt{1-\zeta^2}\, e^{-\zeta \omega_n t}$$