On Numerical Robustness of Constrained RLS-Like Algorithms

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***Abstract*— A number of Constrained RLS–type algorithms have been recently proposed for linearly constrained adaptive applications such as adaptive beamforming. Some of these algorithms, alternatives to employing the Generalized Sidelobe Canceller structure, claim to be robust whether for using a special correction term or for employing numerically stable rotations. Nevertheless, most of these algorithms diverge in long–run simulations or are too sensitive to changes in the forgetting factor. This paper deals with a performance comparison of many Constrained RLS-like algorithms—including two other fast converging non-RLS algorithms—and their GSC counterparts in terms of numerical stability, speed of convergence, and computational complexity. An efficient and recently proposed Householder structure is also taken into account such that the algorithms presenting the most promising results are summarized and appropriate options for fast-converging adaptive beamforming applications accrue.**

I. INTRODUCTION

Adaptation algorithms satisfying linear constraints find application in several areas of signal processing and communications, including beamforming, spectral estimation, and multiuser detection for communication systems. A robust and simple algorithm incorporating the constraints into the solution was first introduced by Frost [1]. This algorithm, like its unconstrained counterpart, suffers from slow convergence when the input signal is highly correlated. As an attempt to overcome this drawback, a number of Constrained Recursive Least Squares (CRLS)-like algorithms have been recently proposed. Because of the well known weak numerical performance of the conventional RLS-based algorithms, some of these new CRLS-based algorithms, alternatives to the use of the Generalized Sidelobe Canceller (GSC) structure [2], use special correction terms [3] or numerically stable rotations [4] to claim robustness. Nevertheless, when running simulations with a considerably large number of samples in the same scenario described in [4], most of these algorithms diverge. It was also observed that the performance of these algorithms is strongly affected by slight variations on the forgetting factor.

This paper deals with the performance of Constrained RLS-like algorithms—we include two other non-RLS but fast-converging algorithms, videlicet, the Constrained Conjugate Gradient (CCG) algorithm [5] and the Constrained *Quasi*-Newton (CQN) algorithm [6]—and their GSC counterparts in terms of speed of convergence, numerical stability, and computational complexity. A Householder structure, recently proposed in [7] as an alternative implementation for Linearly Constrained Minimum Variance (LCMV) filters, is also taken into consideration and the best results are summarized. These results point out the most appropriated options for fast-converging adaptive beamforming applications.

The paper is organized as follows. Section II presents the results of extensive computer simulations showing the convergence and divergence of the severals Constrained RLS-like algorithms. A comparison in terms of speed of convergence and computational complexity, only for those algorithms with good numerical performance, is presented in Section III. In Section IV, some implementation issues of the algorithm attaining the best performance are detailed. Finally, conclusions are summarized in Section V.

II. PERFORMANCE EVALUATION OF THE ALGORITHMS

This section performs an evaluation, in an adaptive beamforming scenario, of the following algorithms: the Constrained Recursive Least Squares (CRLS) [3], the Constrained QRD-RLS (CQRD-RLS) [8], and the Constrained Inverse QRD-RLS (CIQRD-RLS) [4] algorithms, from the RLS family, and two other fast-converging algorithms, the Constrained Conjugate Gradient (CCG) [5] and the Constrained *Quasi*–Newton (CQN) [6] algorithms. Each of the above algorithms has both a GSC and a Householder counterpart which will

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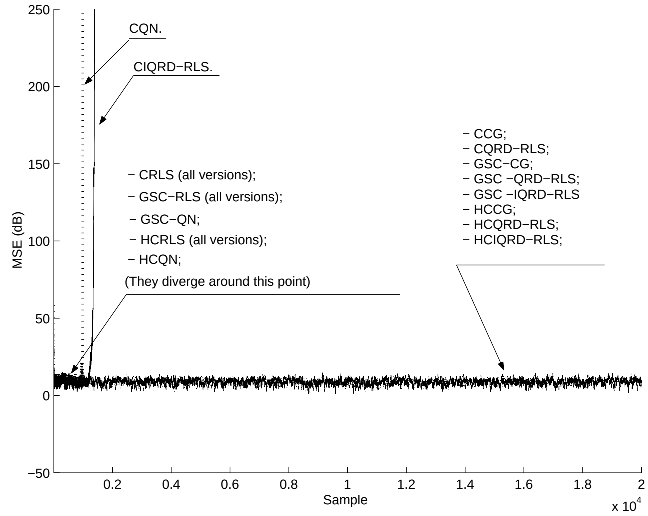

Fig. 1. Convergence of different Constrained, GSC, Householder RLS-like algorithms.

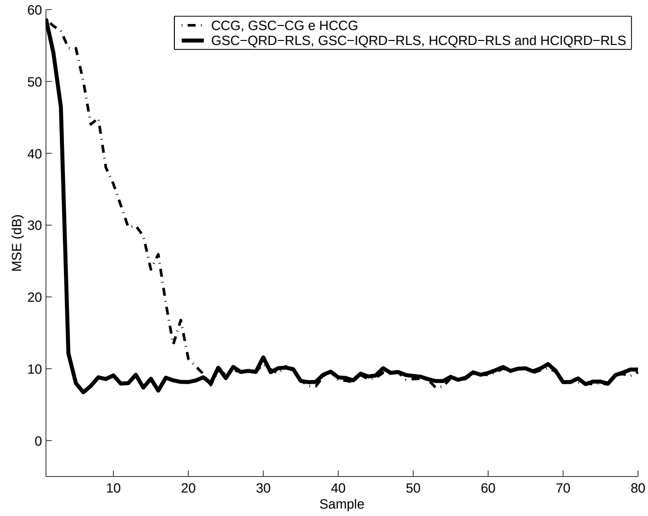

Fig. 2. Learning curve during transient.

be referred to by adding the appropriate prefix (GSC or HC). As an example, for the CRLS algorithm, we have GSC-RLS and HCRLS. We should emphasize that, whether in the GSC or in the Householder structure, one uses an unconstrained version of each algorithm which, in our case, are the conventional RLS algorithm [9], the Conjugate Gradient (CG) algorithm [10], the *Quasi*–Newton (QN) algorithm [11], the conventional QRD-RLS algorithm [9], and the Inverse QRD-RLS (IQRD-RLS) algorithm [12].

The performances of the algorithms are evaluated here concerning their stability; in the following section, their computational complexity and interference rejection capability will be addressed.

In our adaptive beamforming experiment, we have used, as in [4], a linear array of 7 sensors (*isotropic* antennas) with a look-direction set to $0°$ and three *jammers* (interferers) with incident angles corresponding to $-25°$, $45°$, and $50°$. The signal-to-noise ratio (SNR) was set to $0dB$ and jammer-to-noise ratios (JNR) of $30dB$ were used. The forgetting factor (λ) was set to 0.98.

III. Comparing the Performance of the robust algorithms

During the simulations, it has been observed that all versions of the Constrained RLS algorithms presented in [3] are very unstable as well as their GSC and Householder counterparts. Even the Constrained Inverse QRD-RLS [4] algorithm, which was claimed to be numerically more stable than the conventional CRLS algorithm, suffers from numerical instability, in LCMV applications. After 100 runs of 2×10^4 samples each, it can be observed from Fig. 1 that only few algorithms do not diverge: the Constrained Conjugate Gradient algorithm [5], its GSC and Householder Constrained counterparts (using the unconstrained version as described in [10]), the Constrained QRD-RLS algorithm of [8], the GSC and Householder Constrained QRD-RLS [7], [9], and the GSC and Householder Constrained Inverse QRD-RLS [7], [12] algorithms. It is worth mentioning that the Constrained QRD-RLS algorithm of [8], although not diverging, is merely an MVDR (*Minimum Variance Distortionless Response*) that makes its use restricted to this kind of environment, i.e., when there is no reference signal and the gain vector equals to scalar one.

Even though not appearing in Fig. 1, other 10 independent runs of 6×10^5 samples each were carried out and the well-behaved algorithms did not show any sign of divergence.

It is interesting to note that, regardless the value of the forgetting factor (λ), those algorithms presenting diverging behavior always diverged. In the experiment replicated here from [4], we have varied the value of this constant and observed that, sooner or later, all algorithms presenting instability in Fig. 1 diverged, including the Constrained Inverse QRD-RLS algorithm (note that [4] does not mention the value of λ used).

Another remark is that the algorithm introduced in [4] (CIQRD), when compared to the GSC-Inverse QRD-RLS algorithm, does not follow identical learning curves in the first samples (transient), as somehow expected, at least for the case of an orthogonal blocking matrix. Moreover, for those algorithms that converged, their GSC implementations were identical to the Householder Constrained versions as long as $\mathbf{B}^H\mathbf{B} = \mathbf{I}$.

Convergence curves of the numerically robust algo-

TABLE I

COMPUTATIONAL COMPLEXITY OF THE MOST ROBUST ALGORITHMS (COMPLEX OPERATIONS).

ALGORITHM	MULTIP.	DIVIS.	SQRTS
CCG	$MJ(6MJ+2p+8)+1$	1	0
HCCG	$(MJ-p)(5MJ-5p+9)-p(p-2MJ-2)+1$	1	0
HCQRD-RLS	$\frac{(MJ-p)}{2}(13MJ-11p+17)+(2MJ-p+2)p+1$	$2(MJ-p)$	$MJ-p$
HCIQRD-RLS	$\frac{(MJ-p)^3}{2}+(MJ-p)(5MJ-5p+7)+p(2MJ-p+1)+2$	$2(MJ-p)+1$	$MJ-p$
GSC-CG	$MJ(7MJ-12p+9)+p(5p-8)+1$	1	0
GSC-QRD-RLS	$(MJ-p)(6MJ-5p+9)+MJ+1$	$2(MJ-p)$	$MJ-p$
GSC-IQRD-RLS	$\frac{(MJ-p)^3}{2}+(MJ-p)(6MJ-5p+7)+MJ+2$	$2(MJ-p)+1$	$MJ-p$

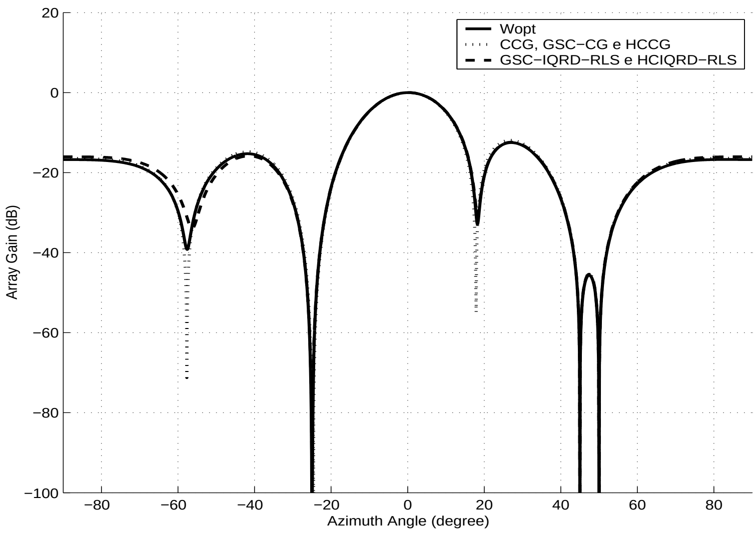

Fig. 3. Beam patterns for the case of 20 iterations and one single run.

rithms are depicted in Fig. 2 for the first 80 samples. This figure clearly shows the fast convergence rate of the QRD-RLS, the IQRD-RLS, and the Constrained Conjugate Gradient algorithms, along with their GSC and Householder counterparts, for this experiment.

The computational complexity of the converging algorithms is summarized in Table 1, where M and J stand for the number of channels and the number of *taps* of the filter coefficients, respectively, and p is the number of constraints. In order to exemplify the computational burden for the particular case of the simulations actually carried out for this paper, Table 2 presents the results for $M = 7$, $J = 1$, and $p = 1$.

From these tables, it can be seen that, at least for this application, the HCCG algorithm presents a good performance. This algorithm not just attains to minimum complexity but also, as can be observed in Fig. 3, along with the CCG, the GSC-CG, the QRD-RLS, and the IQRD-RLS based algorithms, its beam pattern matches the optimal quite fast, meaning a good *sample support* capability.

TABLE II

COMPUTATIONAL COMPLEXITY (NUMERICAL EXAMPLE).

ALGORITHM	MULTIP.	DIVIS.	SQRTS
CCG	365	1	0
HCCG	**249**	**1**	**0**
HCQRD-RLS	250	12	6
HCIQRD-RLS	346	13	6
GSC-CG	284	1	0
GSC-QRD-RLS	284	12	6
GSC-IQRD-RLS	381	13	6

IV. IMPLEMENTATION ISSUES OF THE HCCG ALGORITHM

In this section, we provide some details of the algorithm with the lowest computational complexity as found in the former section. This algorithm, although never

TABLE III
THE HCCG ALGORITHM.

Available at time instant k:
  $\mathbf{x}(k)$, $\mathbf{C}$ $\mathbf{f}$, and $\mathbf{Q}$
Initialize:
  λ, η with $(\lambda - 0.5) \leq \eta \leq \lambda$
  δ small number
  $\overline{\mathbf{w}}(0) = \mathbf{QF} = \mathbf{QC}(\mathbf{C^H C})^{-1}\mathbf{f}$
  $\overline{\mathbf{R}} = \mathbf{I}_{MJ-p}$
  $\mathbf{g}(0) = \mathbf{c}(0) = zeros(MJ - p, 1)$
for each k
{
  $\overline{\mathbf{x}}(k) = \mathbf{Q}\mathbf{x}(k)$;
  $\overline{\mathbf{x}}_U(k) = p$ first elements of $\overline{\mathbf{x}}(k)$;
  $\overline{\mathbf{x}}_L(k) = MJ - p$ last elements of $\overline{\mathbf{x}}(k)$;
  $\overline{\mathbf{w}}(k) = \begin{bmatrix} \overline{\mathbf{w}}_U(0) \\ -\overline{\mathbf{w}}_L(k-1) \end{bmatrix}$;
  $\overline{\mathbf{R}}(k) = \lambda\overline{\mathbf{R}}(k-1) + \overline{\mathbf{x}}_L(k)\overline{\mathbf{x}}_L^H(k)$
  $\overline{e}(k) = \overline{\mathbf{w}}_U^H(0)\overline{\mathbf{x}}_U(k) - \overline{\mathbf{w}}_L^H(k-1)\overline{\mathbf{x}}_L(k)$;
  $\alpha(k) = \eta\frac{\mathbf{c}^H(k)\mathbf{g}(k-1)}{\mathbf{c}^H(k)\overline{\mathbf{R}}(k)\mathbf{c}(k)+\delta}$
  $\mathbf{g}(k) = \lambda\mathbf{g}(k-1) - \alpha(k)\overline{\mathbf{R}}(k)\mathbf{c}(k) - \mathbf{x}_L(k)\overline{e}^*(k)$;
  $\overline{\mathbf{w}}_L(k) = \overline{\mathbf{w}}_L(k-1) - \alpha\mathbf{c}(k)$
  $\beta(k) = \frac{[\mathbf{g}(k)-\mathbf{g}(k-1)]^H\mathbf{g}(k)}{\mathbf{g}^H(k-1)\mathbf{g}(k-1)+\delta}$;
  $\mathbf{c}(k+1) = \mathbf{g}(k) + \beta(k)\mathbf{c}(k)$;
}

been presented before as a formal algorithm, is actually a combination of the techniques used in [5] and [7].

The HCCG algorithm is detailed in Table 3. Here, $\mathbf{C}_{MJ\times p}$ is the constraint matrix and $\mathbf{f}_{p\times 1}$ is the gain vector. $\mathbf{Q}$ is an orthogonal rotation matrix, constructed with successive Householder transformations, and it is used to generate the transformed coefficient vector $\overline{\mathbf{w}}(k)$ such that $\overline{\mathbf{w}}(k) = \mathbf{Q}\mathbf{w}(k)$. For more details on how to compute properly matrix $\mathbf{Q}$ and how to multiply efficiently this matrix by the input vector, the reader is encouraged to read [7].

V. CONCLUSIONS

In this paper, the performance in a typical LCMV application of several published versions of the Constrained RLS-based algorithms were evaluated. We included those that use numerically stable rotations and two other fast converging constrained algorithms. After tests with a large number of samples, we have concluded that only the CCG and the QRD-RLS (both conventional and inverse versions) when used under special structures (GSC and HC), have acceptable performance with respect to robustness. This leads to the conclusion that the use of structures like the GSC and the Householder are the only viable option so far for Constrained RLS adaptive filtering.

Among the assessed algorithms, the one that had a stable performance and a fast convergence in a direct constrained version was the CCG algorithm.

The Householder Constrained Conjugate Gradient algorithm, for its low computational complexity, was considered the most suitable option for the particular adaptive beamforming application implemented. When used with GSC or Householder structures, the CG, the QRD-RLS, and the IQRD-RLS algorithms showed improved robustness and were considered workable alternative implementations.

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