Microphone-Array Signal Processing

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IME – Lab. Processamento Digital de Sinais

Outline

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1. Introduction and Fundamentals

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2. Sensor Arrays and Spatial Filtering

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3. Optimal Beamforming

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3. Optimal Beamforming
4. Adaptive Beamforming
5. DoA Estimation with Microphone Arrays

1. Introduction and Fundamentals

General concepts

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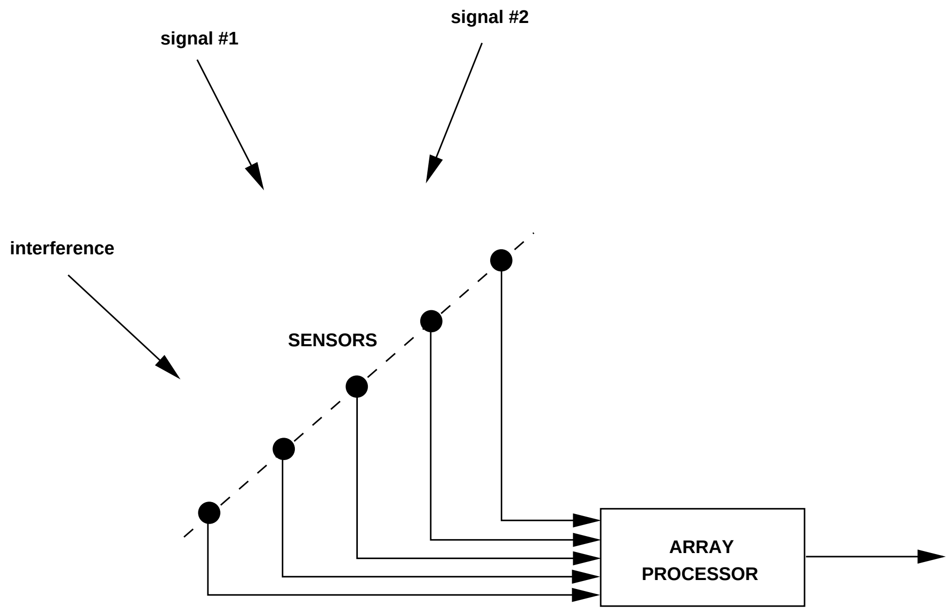

1.2 Signals in Space and Time

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and

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respectively.

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$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2}$$

where: c is the propagation speed, $\overrightarrow{E}$ is the electric field intensity, and $\mathbf{x} = [x\ y\ z]^T$ is a position vector.

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Note: From this point onwards the terms *wave* and *field* will be used interchangeably.

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$$s(\mathbf{x}, t) = Ae^{j(\omega t - k_x x - k_y y - k_z z)}$$

where A is a complex constant and k_x, k_y, k_z, and $\omega \geq 0$ are real constants.

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constraints to be satisfied
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$$k_x x + k_y y + k_z z = C$$

where C is a constant.

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The planes where $s(\mathbf{x}, t)$ is constant are perpendicular to the wavenumber vector $\mathbf{k}$

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Remember the constraints?
$\|\mathbf{k}\|^2 = \omega^2/c^2$

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The wavenumber vector, $\mathbf{k}$, may be considered a *spatial frequency* variable, just as ω is a *temporal frequency* variable.

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We may rewrite the wave equation as

$$\begin{aligned} s(\mathbf{x}, t) &= Ae^{j(\omega t - \mathbf{k}^T \mathbf{x})} \\ &= Ae^{j\omega(t - \boldsymbol{\alpha}^T \mathbf{x})} \end{aligned}$$

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As $c = \omega/\|\mathbf{k}\|$, vector $\boldsymbol{\alpha}$ has a magnitude which is the reciprocal of c.

Periodic propagating periodic waves

Any arbitrary periodic waveform $s(\mathbf{x}, t) = s(t - \boldsymbol{\alpha}^T \mathbf{x})$ with fundamental period ω_0 can be represented as a sum:

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We will come back to this later...

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2. Sensor Arrays and Spatial Filtering

2.1 Wavenumber-Frequency Space

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spatial frequency: wavenumber

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$$s(\mathbf{x}, t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\mathbf{k}, \omega) e^{j(\omega t - \mathbf{k}^T \mathbf{x})} d\mathbf{k}\, d\omega$$

Space-time Fourier Transform

We have already concluded that if the space-time signal is a propagating waveform such that $s(\mathbf{x}, t) = s(t - \boldsymbol{\alpha}_0^T \mathbf{x})$, then its Fourier transform is equal to

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Remember the nonperiodic propagating wave Fourier transform?

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Remember the nonperiodic propagating wave Fourier transform?

This means that $s(\mathbf{x}, t)$ only has energy along the direction of $\mathbf{k} = \mathbf{k}_0 = \omega\boldsymbol{\alpha}_0$ in the wavenumber-frequency space.

2.2 Frequency-Wavenumber (WN) Response and Beam patterns (BP)

Signals at the sensors

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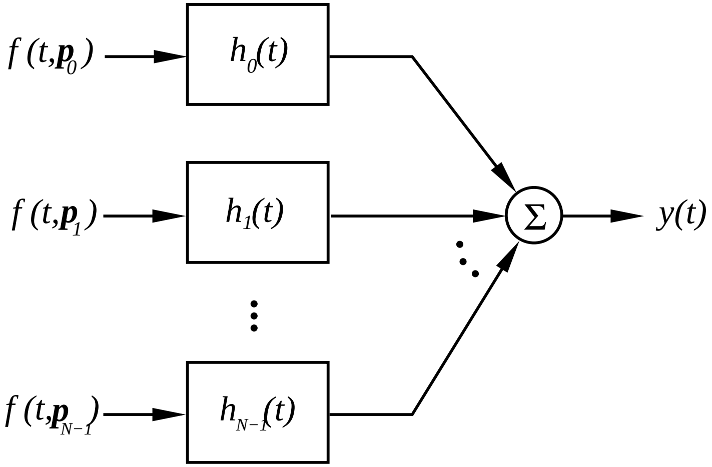
$$\boldsymbol{f}(t, \boldsymbol{p}) = \begin{bmatrix} f(t, \boldsymbol{p}_0) \\ f(t, \boldsymbol{p}_1) \\ \vdots \\ f(t, \boldsymbol{p}_{N-1}) \end{bmatrix}$$

Array output



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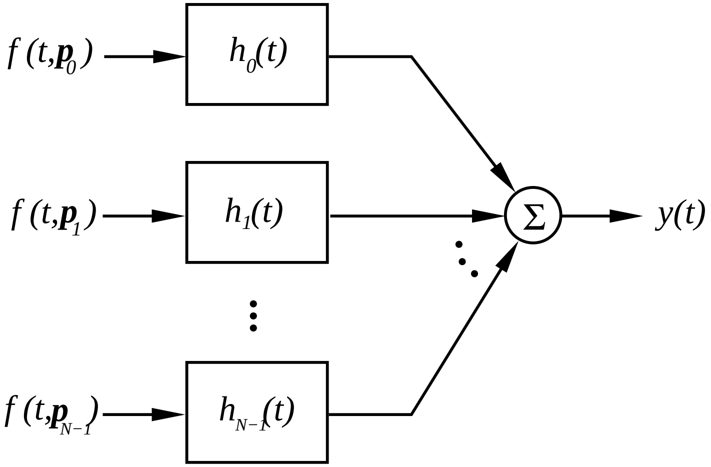

$$y(t) = \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} h_n(t-\tau) f_n(\tau, \boldsymbol{p}_n) d\tau$$
$$= \int_{-\infty}^{\infty} \boldsymbol{h}^T(t-\tau) \boldsymbol{f}(\tau, \boldsymbol{p}) d\tau$$

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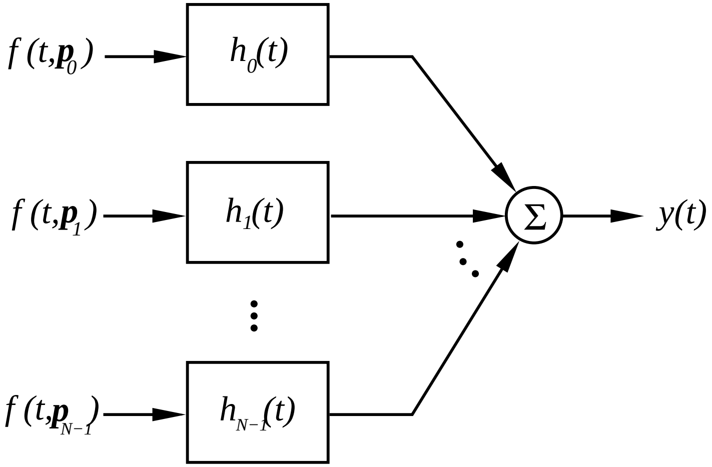

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where $\boldsymbol{h}(t) = [h_o(t) \;\; h_1(t) \;\; \cdots \;\; h_{N-1}(t)]^T$

In the frequency domain, $\cdots$

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Plane wave propagating $\cdots$

- Consider a plane wave propagating in the direction of vector $\boldsymbol{a}$:

$$\boldsymbol{a} = \begin{bmatrix} -sin\theta cos\phi \\ -sin\theta sin\phi \\ -cos\theta \end{bmatrix}$$

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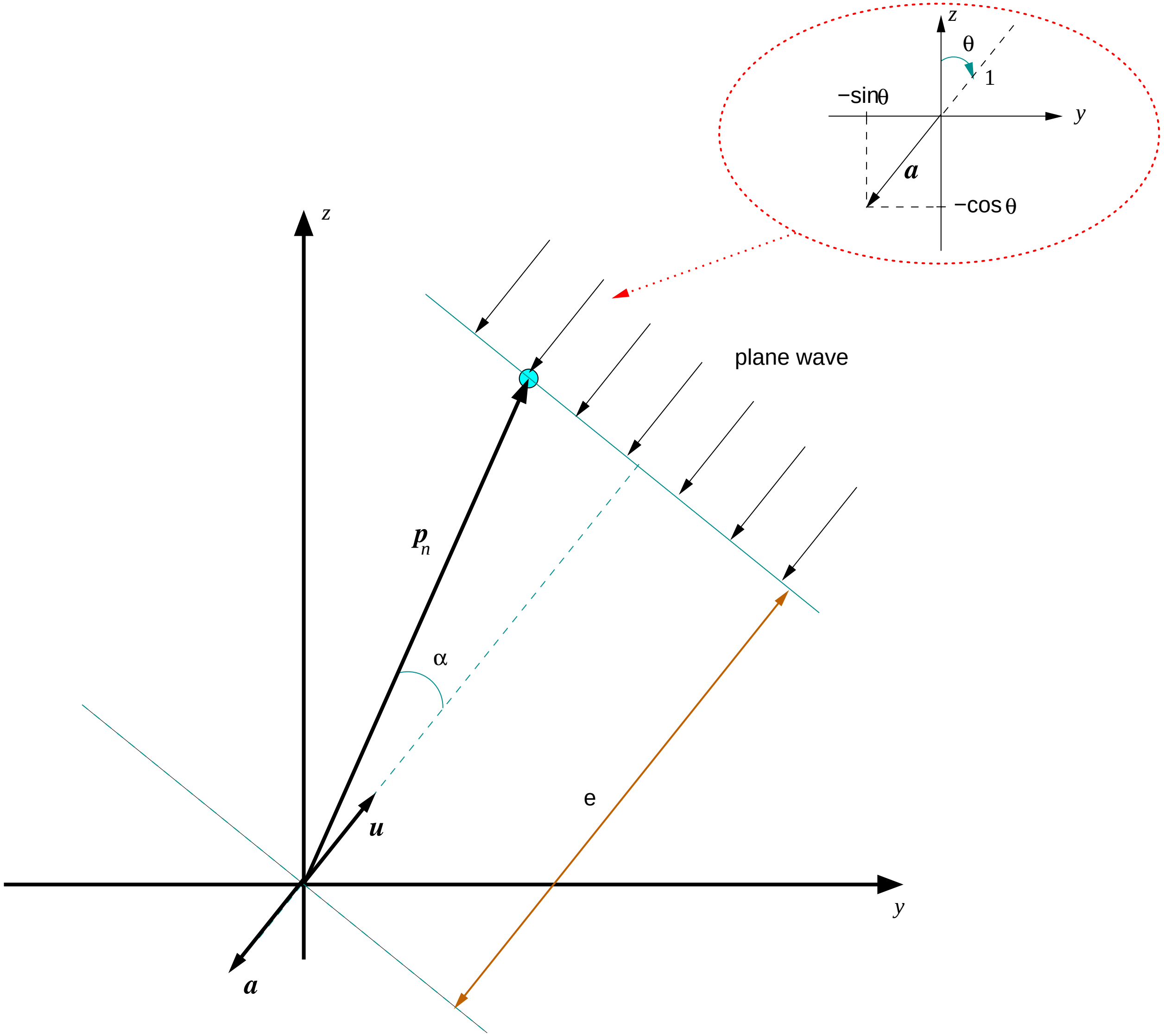
- If $f(t)$ is the signal that would be received at the origin, then:

$$\boldsymbol{f}(t, \boldsymbol{p}) = \begin{bmatrix} f(t-\tau_0) \\ f(t-\tau_1) \\ \vdots \\ f(t-\tau_{N-1}) \end{bmatrix}$$

Plane wave (assuming $\phi = 90^o$)



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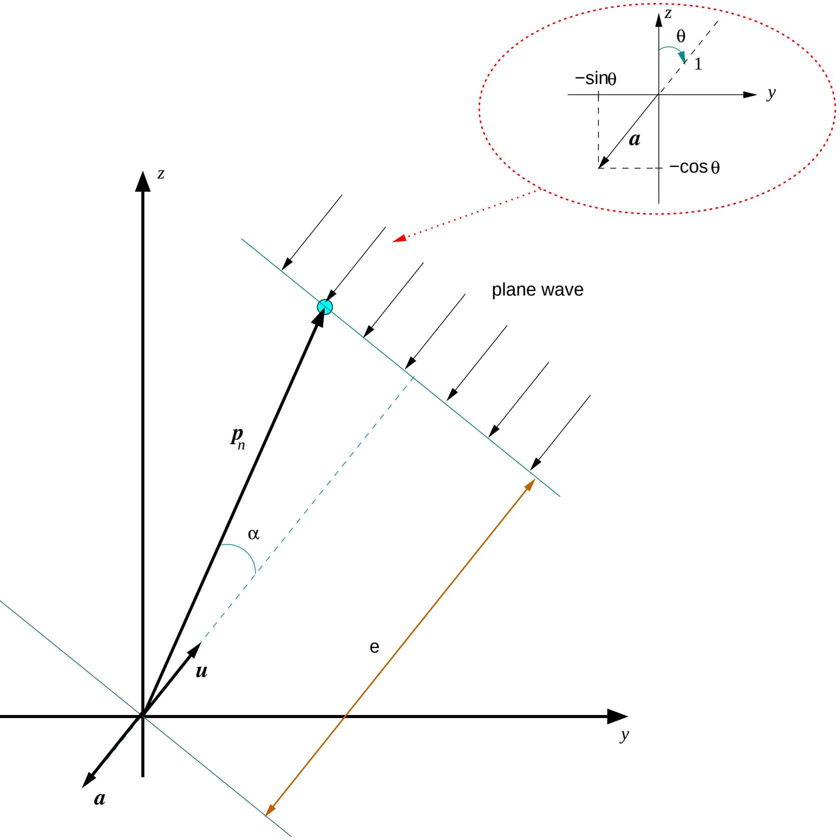

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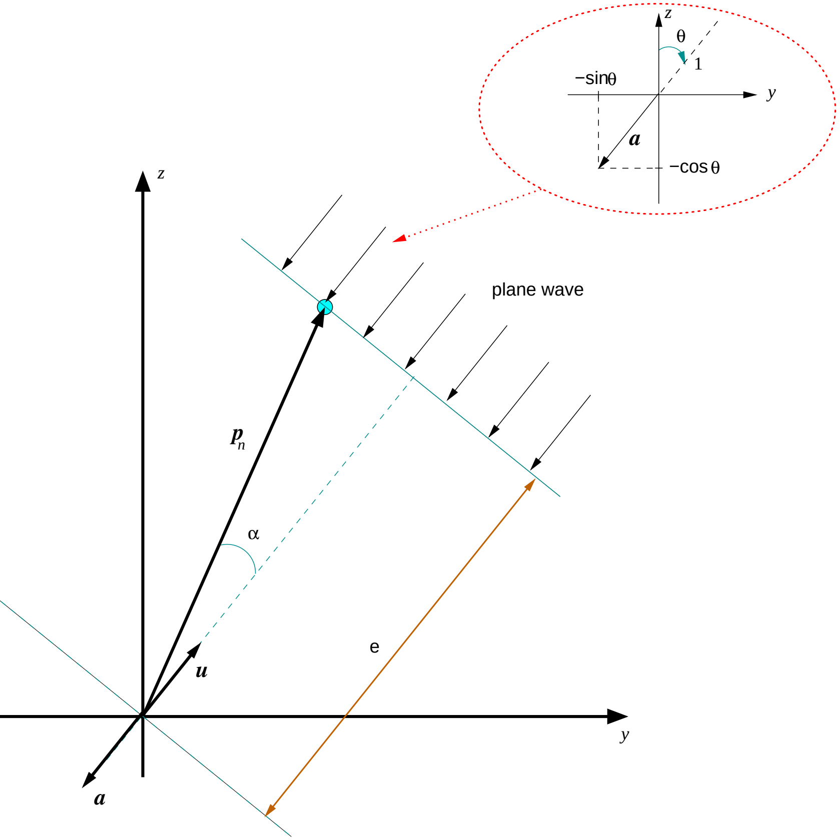

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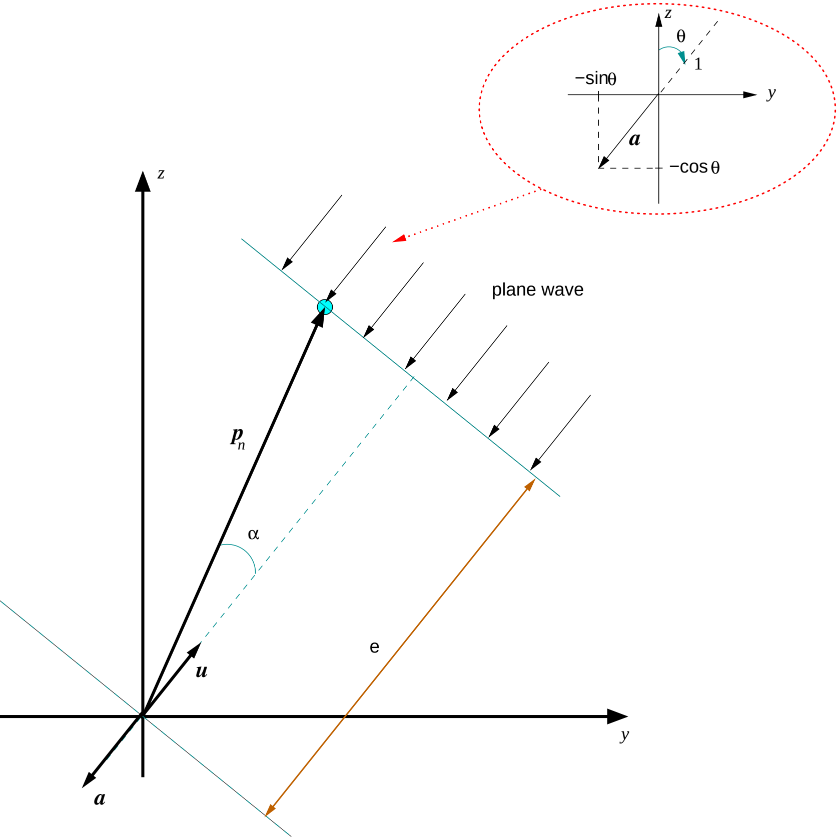

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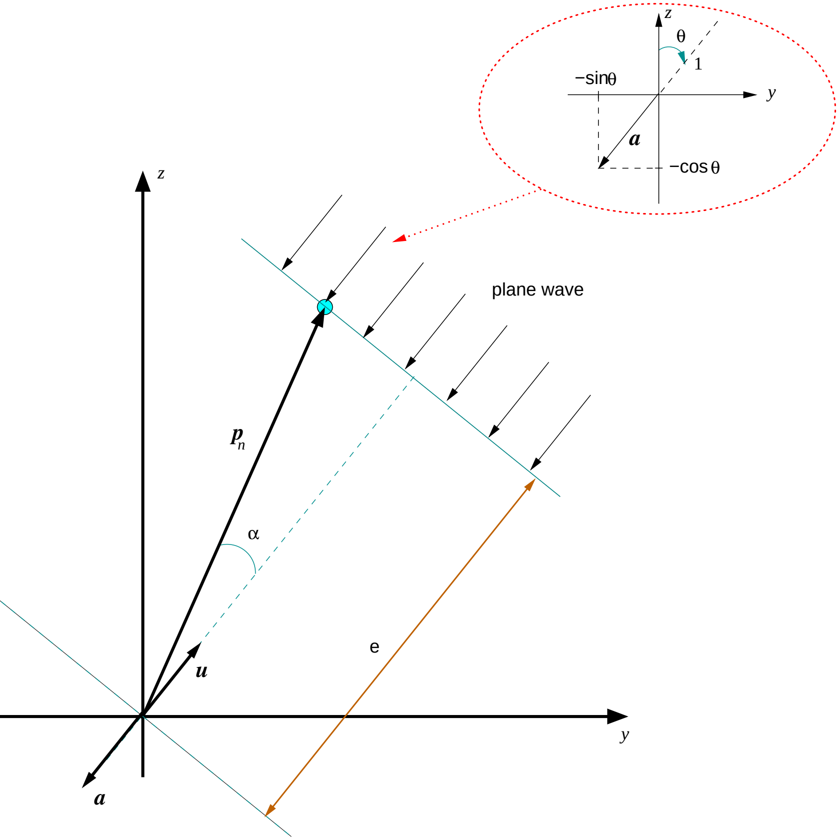

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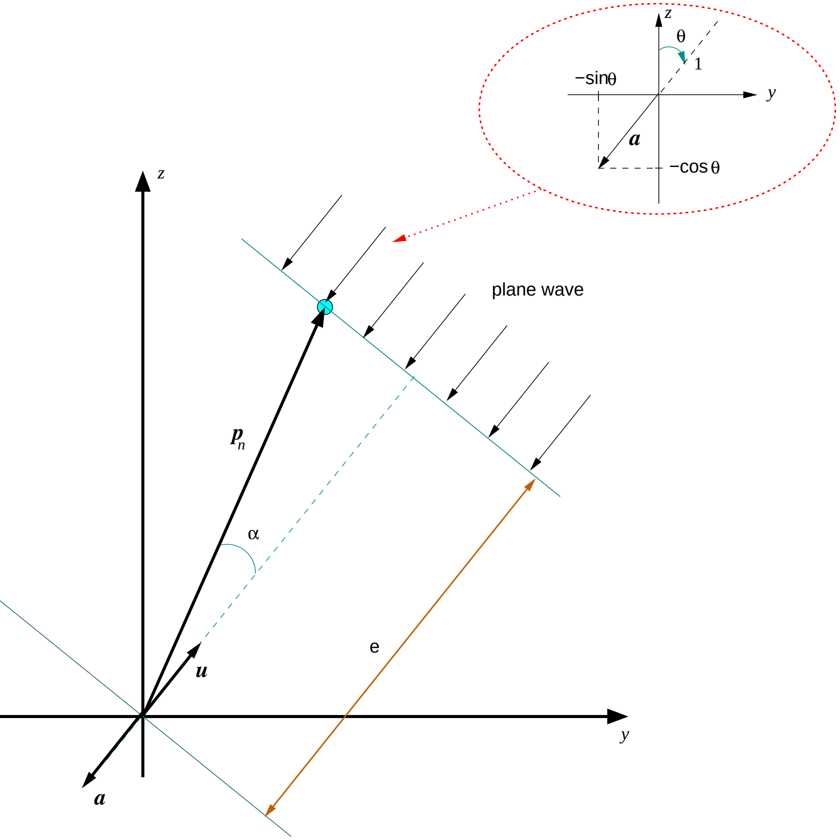

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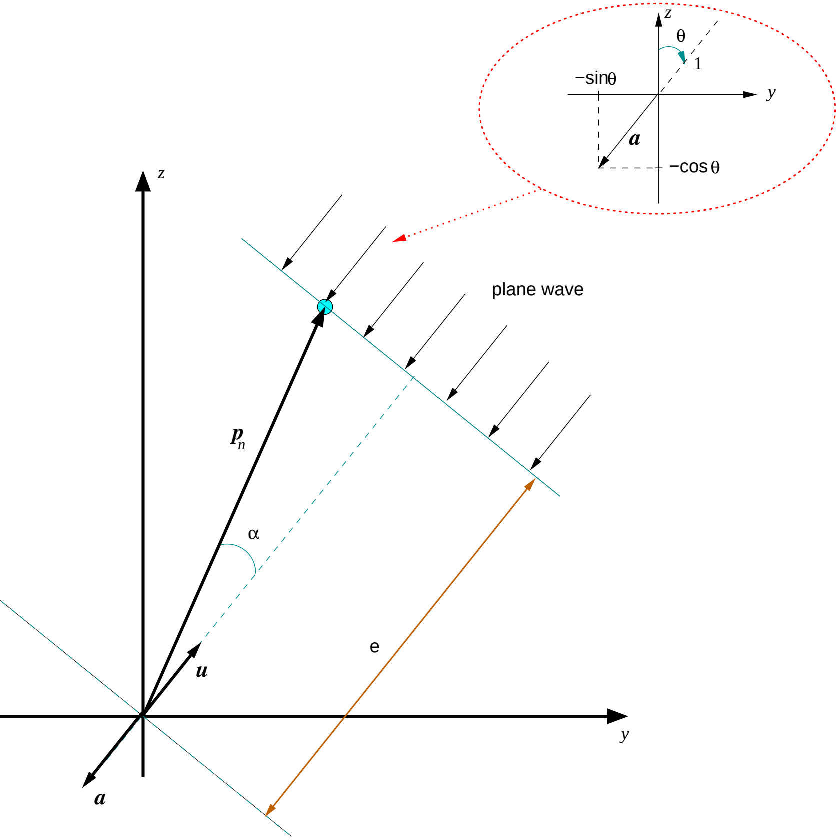

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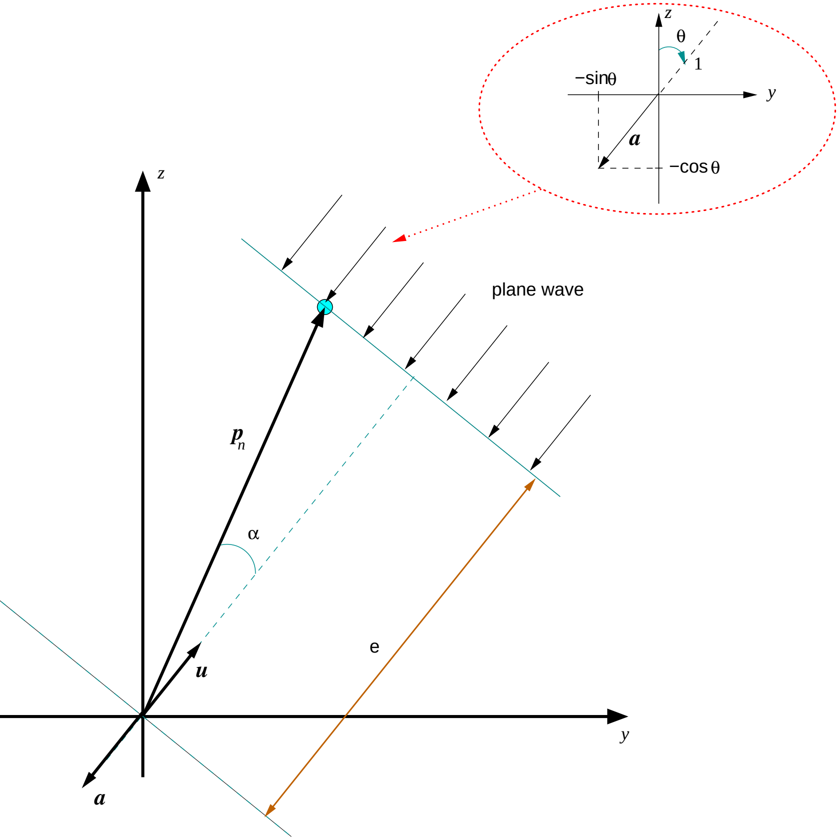

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$_n$ is the time since the plane wave hits the sensor at location $\mathbf{p}_n$ until it reaches point $(0,0)$.

Back to the frequency domain

- Then, we have:

$$\boldsymbol{F}(\omega) = \begin{bmatrix} \int_{-\infty}^{\infty} e^{-j\omega t} f(t-\tau_0)dt \\ \int_{-\infty}^{\infty} e^{-j\omega t} f(t-\tau_1)dt \\ \vdots \\ \int_{-\infty}^{\infty} e^{-j\omega t} f(t-\tau_{N-1})dt \end{bmatrix}$$

$$= \begin{bmatrix} e^{-j\omega\tau_0} \\ e^{-j\omega\tau_1} \\ \vdots \\ e^{-j\omega\tau_{N-1}} \end{bmatrix} F(\omega)$$

Definition of Wavenumber

- For plane waves propagating in a locally homogeneous medium:

$$\boldsymbol{k} = \frac{\omega}{c}\boldsymbol{a} = \frac{2\pi}{c/f}\boldsymbol{a} = \frac{2\pi}{\lambda}\boldsymbol{a} = -\frac{2\pi}{\lambda}\boldsymbol{u}$$

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$$\omega\tau_n = \frac{\omega}{c}\boldsymbol{a}^T\boldsymbol{p}_n = \boldsymbol{k}^T\boldsymbol{p}_n$$

Array Manifold Vector

- And we have

$$\boldsymbol{F}(\omega) = \begin{bmatrix} e^{-j\boldsymbol{k}^T\boldsymbol{p}_0} \\ e^{-j\boldsymbol{k}^T\boldsymbol{p}_1} \\ \vdots \\ e^{-j\boldsymbol{k}^T\boldsymbol{p}_{N-1}} \end{bmatrix} F(\omega) = F(\omega)\boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k})$$

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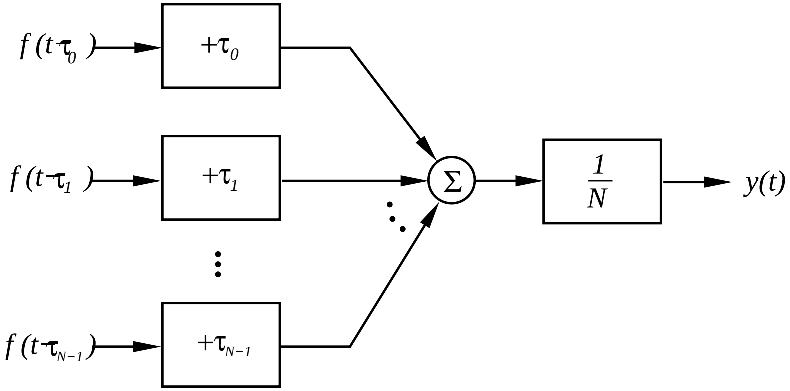
- In this particular example, we can use $h_n(t) = \frac{1}{N}\delta(t + \tau_n)$ such that

$$y(t) = f(t)$$

  Following, we have the delay-and-sum beamformer.

Delay-and-sum Beamformer



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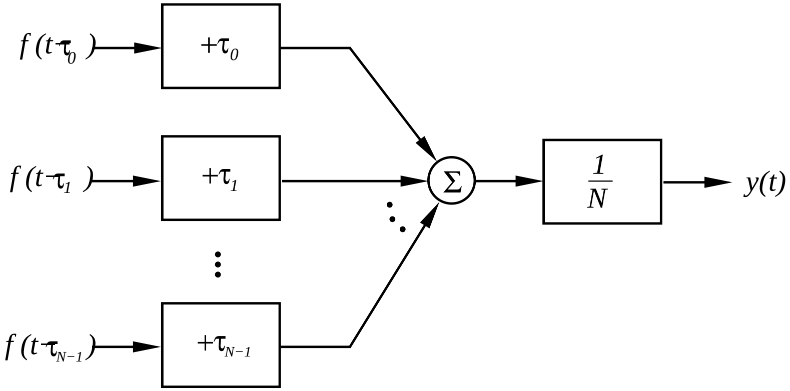

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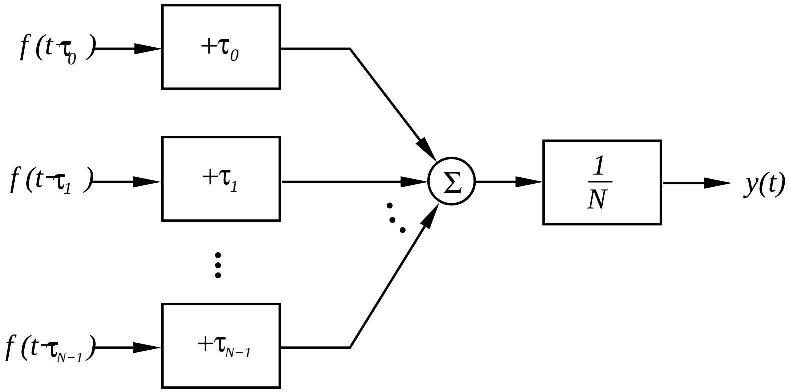

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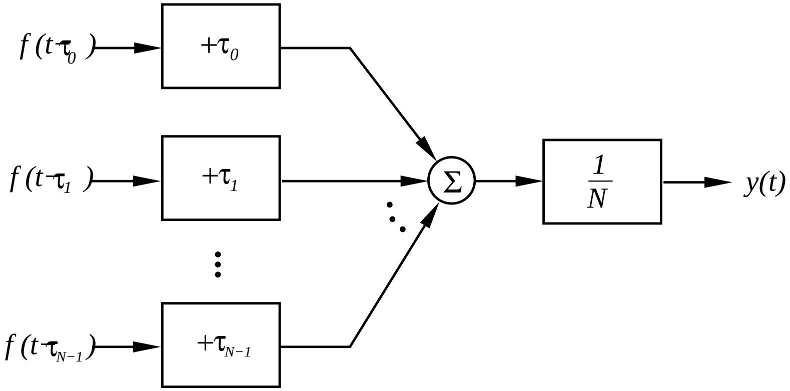

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$$\boldsymbol{H}^T(\omega) = \frac{1}{N}\boldsymbol{v}_{\boldsymbol{k}}^H(\boldsymbol{k})$$

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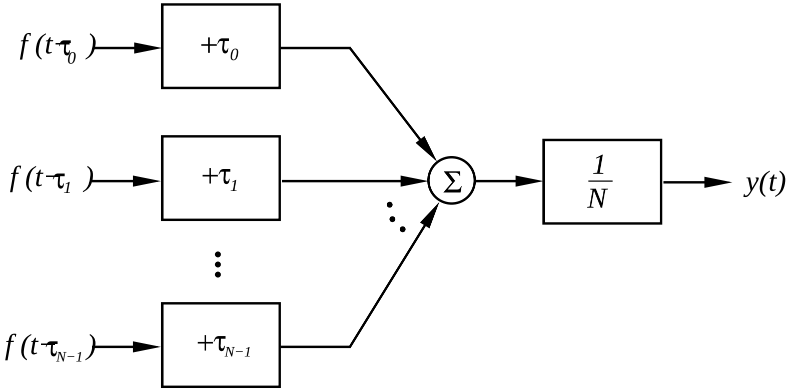

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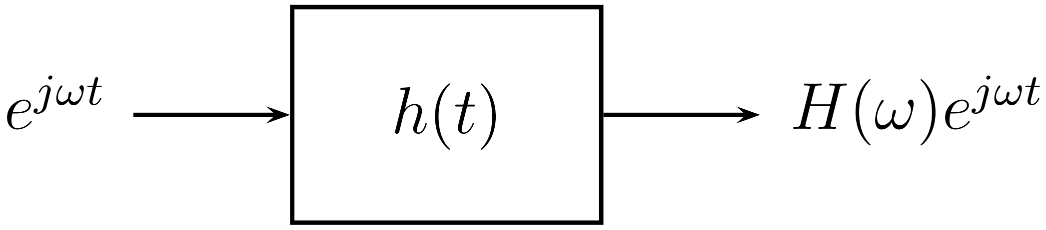

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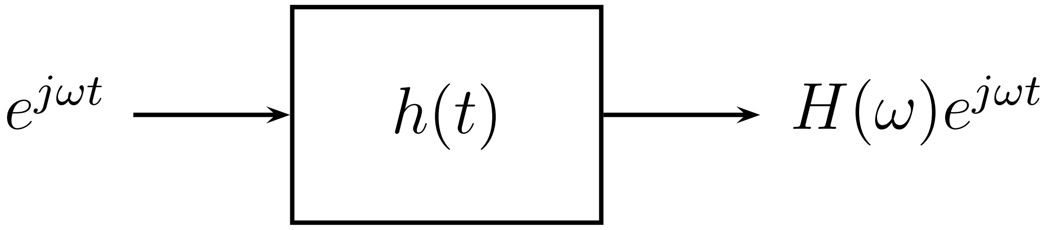

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(\omega)e^{j\omega t}$$

- Space-time signals (base functions):

$$f_n(t, \boldsymbol{p}) = e^{j\omega(t-\tau_n)} = e^{j(\omega t - \boldsymbol{k}^T\boldsymbol{p}_n)}$$

Note that $\omega\tau_n = \boldsymbol{k}^T\boldsymbol{p}_n$

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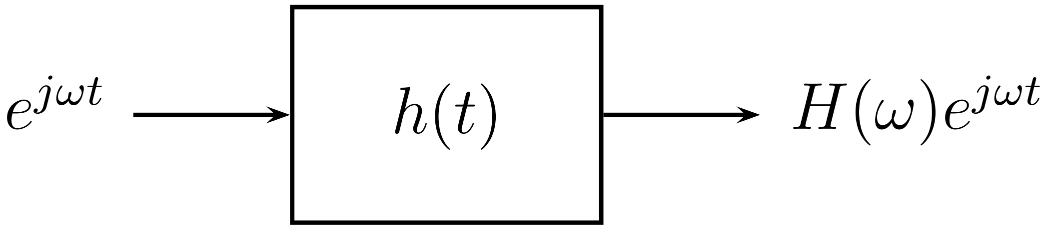

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- The response of the array to this plane wave is:

$$y(t, \boldsymbol{k}) = \boldsymbol{H}^T(\omega)\boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k})e^{j\omega t}$$

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- And we define the Frequency-Wavenumber Response Function:

Upsilon

$$\Upsilon(\omega, \boldsymbol{k}) \triangleq \boldsymbol{H}^T(\omega)\boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k})$$

$\Upsilon(\omega, \boldsymbol{k})$ describes the complex gain of an array to an input plane wave with wavenumber $\boldsymbol{k}$ and temporal frequency ω.

Beam Pattern and Bandpass Signal

- BEAM PATTERN is the Frequency Wavenumber Response Function evaluated versus the direction:

$$B(\omega : \theta, \phi) = \Upsilon(\omega, \boldsymbol{k})$$

Note that $\boldsymbol{k} = \frac{2\pi}{\lambda}\boldsymbol{a}(\theta, \phi)$, and $\boldsymbol{a}$ is the unit vector with spherical coordinates angles θ and ϕ

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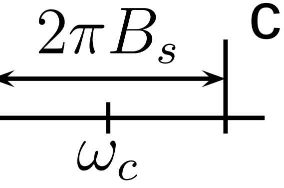
- ω_c corresponds to the carrier frequency and the complex envelope $\tilde{f}(t, \boldsymbol{p}_n)$ is bandlimited to the region



$$|\underbrace{\omega - \omega_c}_{\omega_L}| \leq 2\pi B_s/2$$

Bandlimited and Narrowband Signals

- Bandlimited plane wave:

$$f(t, \boldsymbol{p}_n) = \sqrt{2} Re\{\tilde{f}(t - \tau_n) e^{j\omega_c (t - \tau_n)}\}, n = 0, 1, \cdots, N - 1$$

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- In Narrowband (NB) signals, $B_s \Delta T_{max} \ll 1$
  $\Rightarrow \tilde{f}(t - \tau_n) \simeq \tilde{f}(t)$ and
  $f(t, \boldsymbol{p}_n) = \sqrt{2} Re\{\tilde{f}(t) e^{-j\omega_c \tau_n} e^{j\omega_c t}\}$

Phased-Array

- For NB signals, the delay is approximated by a phase-shift:
  $\Rightarrow$ delay&sum beamformer $\equiv$ PHASED ARRAY

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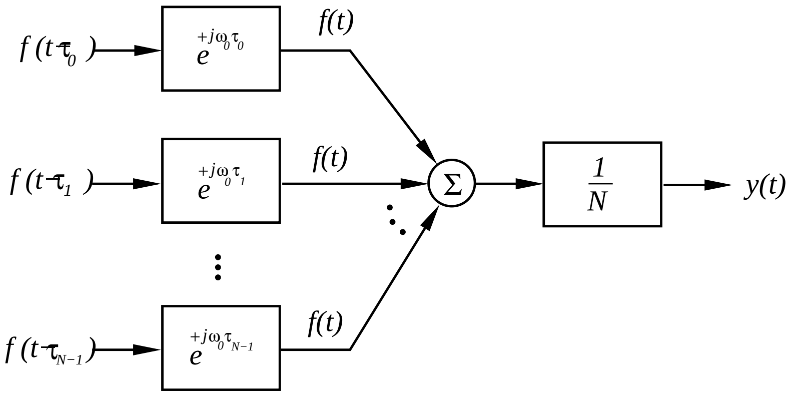

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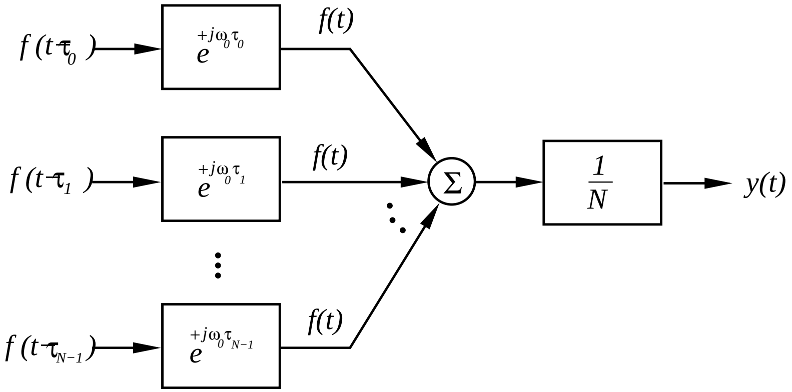

- The phased array can be implemented adjusting the gain and phase to achieve a desired beam pattern

NB Beamformers

- In narrowband beamformers: $y(t, \boldsymbol{k}) = \boldsymbol{w}^H \boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k}) e^{j\omega t}$

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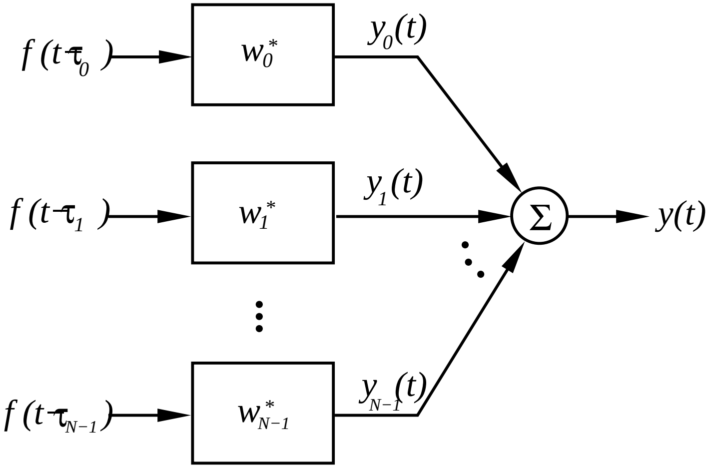

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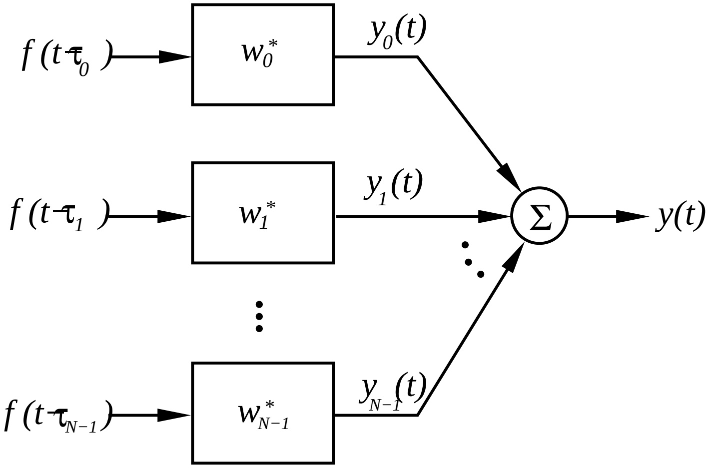
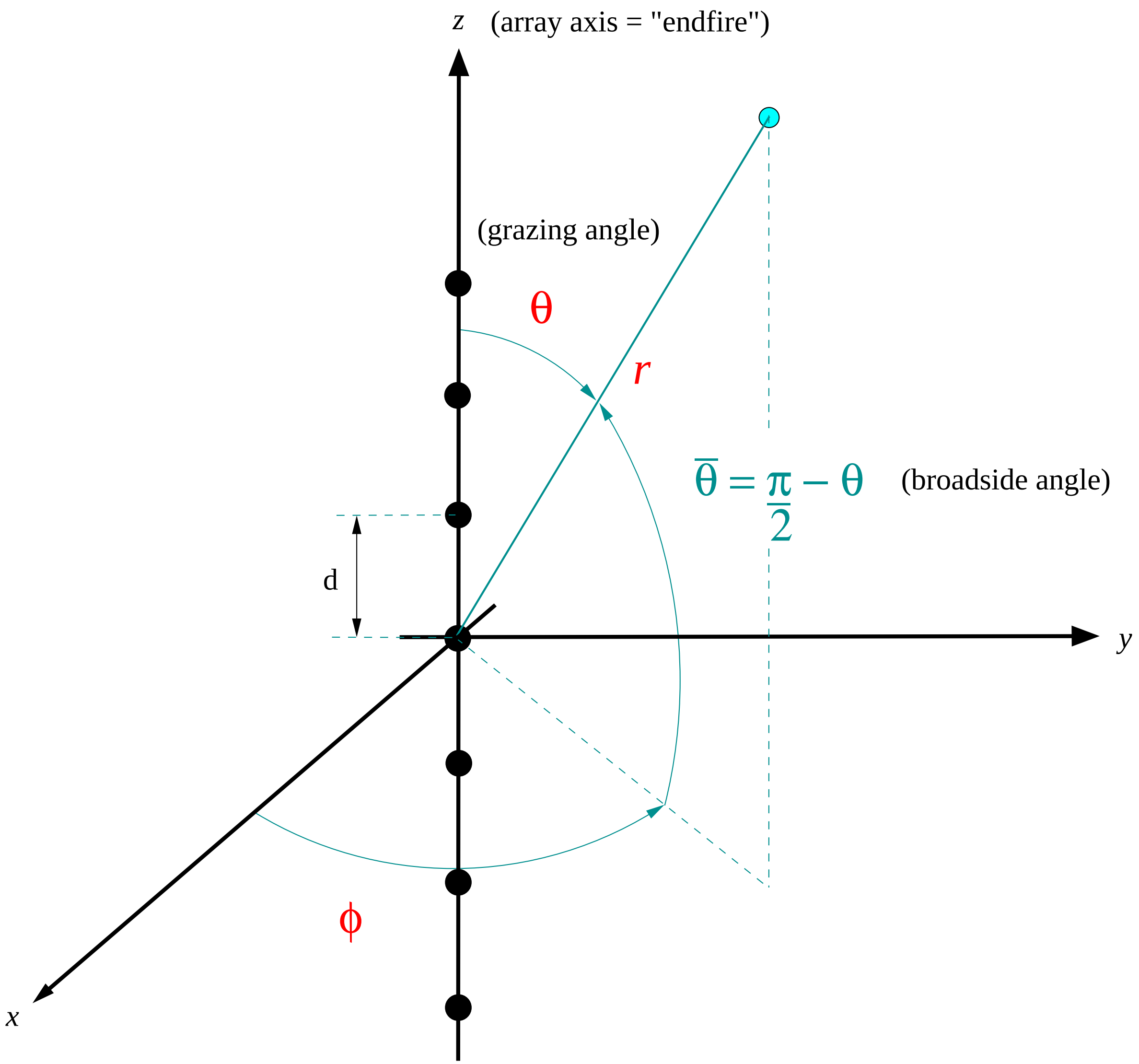


- $\Upsilon(\omega, \boldsymbol{k}) = \underbrace{\boldsymbol{w}^H}_{\boldsymbol{H}^T(\omega)} \boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k})$

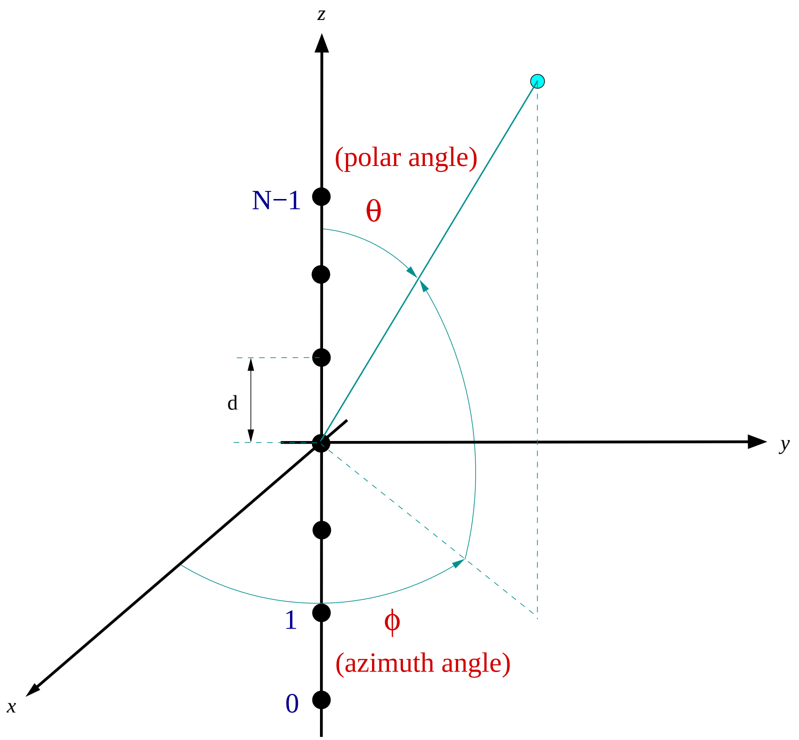
2.3 Uniform Linear Arrays (ULA)

Uniformly Spaced Linear Arrays



ULA

- An ULA along axis z:



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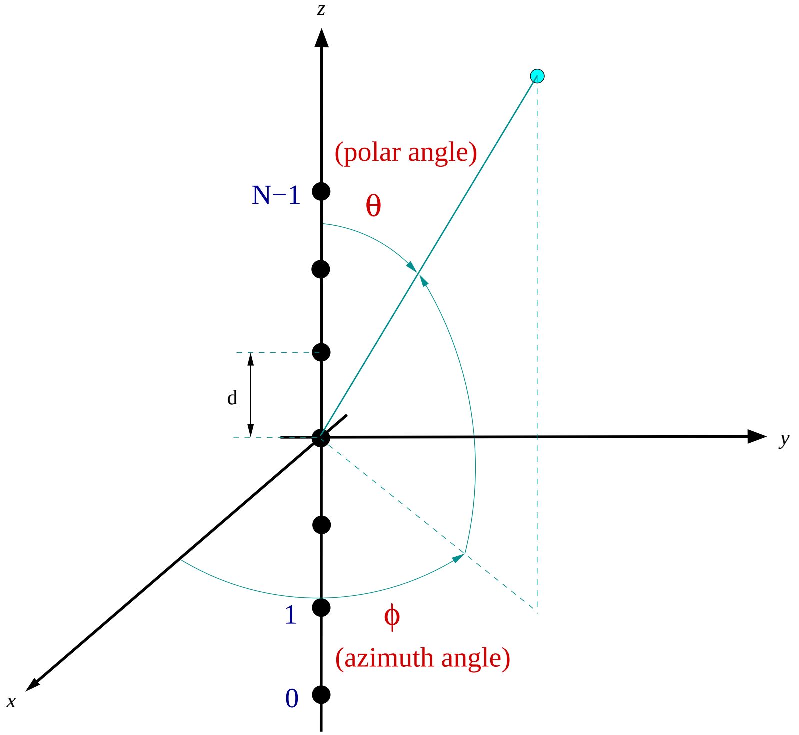

- Location of the elements:

$$\begin{cases} p_{zn} = (n - \frac{N-1}{2})d, \text{ for } n = 0, 1, \cdots, N-1 \\ p_{xn} = p_{yn} = 0 \end{cases}$$

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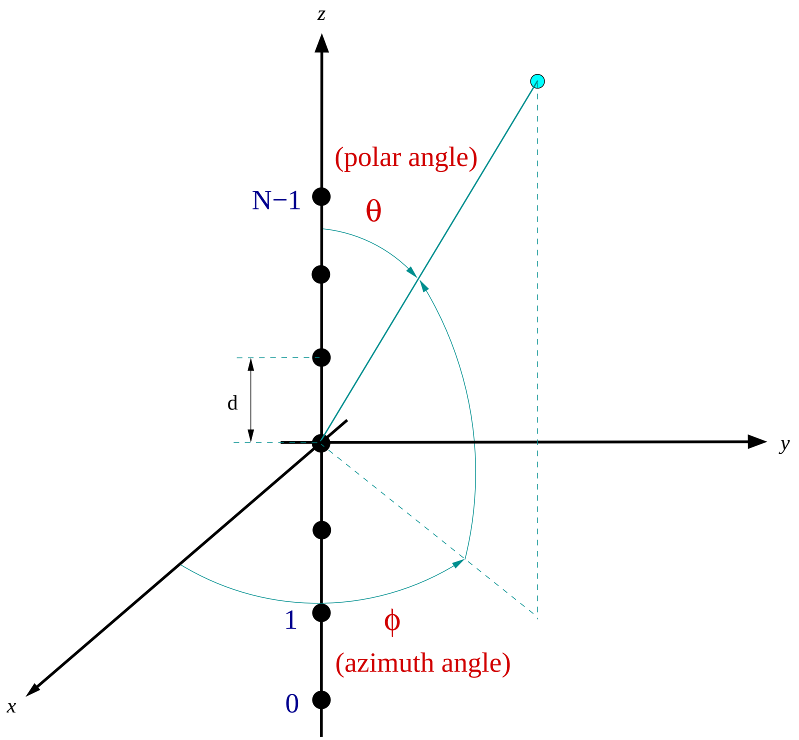

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$$\begin{cases} p_{zn} = (n - \frac{N-1}{2})d, & \text{for } n = 0, 1, \cdots, N-1 \\ p_{xn} = p_{yn} = 0 \end{cases}$$

- Therefore, $\boldsymbol{p}_n = \begin{bmatrix} 0 \\ 0 \\ (n - \frac{N-1}{2})d \end{bmatrix}$

ULA

- Array manifold vector:

$$\boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k}) = \begin{bmatrix} e^{-j\boldsymbol{k}^T\boldsymbol{p}_0} \\ \vdots \\ e^{-j\boldsymbol{k}^T\boldsymbol{p}_n} \\ \vdots \\ e^{-j\boldsymbol{k}^T\boldsymbol{p}_{N-1}} \end{bmatrix}$$

$$\begin{bmatrix} k_x & k_y & k_z \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \left[n - \frac{N-1}{2}\right] d \end{bmatrix}$$

ULA

- Array manifold vector:

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$$\therefore \boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k}) = \boldsymbol{v}_{\boldsymbol{k}}(k_z) = \begin{bmatrix} e^{+j\frac{(N-1)}{2}k_z d} \\ e^{+j(\frac{N-1}{2}-1)k_z d} \\ \vdots \\ e^{-j(\frac{N-1}{2})k_z d} \end{bmatrix}$$

Outline

1. Introduction and Fundamentals
2. Sensor Arrays and Spatial Filtering
3. Optimal Beamforming
4. Adaptive Beamforming
5. DoA Estimation with Microphone Arrays

3. Optimal Beamforming

3.1 Introduction

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- Scope: use the statistical representation of signal and noise to design array processors that are optimal in a statistical sense.

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- Our objective of interest is to estimate the waveform of a plane-wave impinging on the array in the presence of noise and interfering signals.
- Even if a particular beamformer developed in this chapter has good performance, it does not guarantee that its adaptive version (next chapter) will. However, if the performance is poor, it is unlikely that the adaptive version will be useful.

3.2 Optimal Beamformers

MVDR Beamformer

Snapshot model in the frequency domain:

MVDR Beamformer

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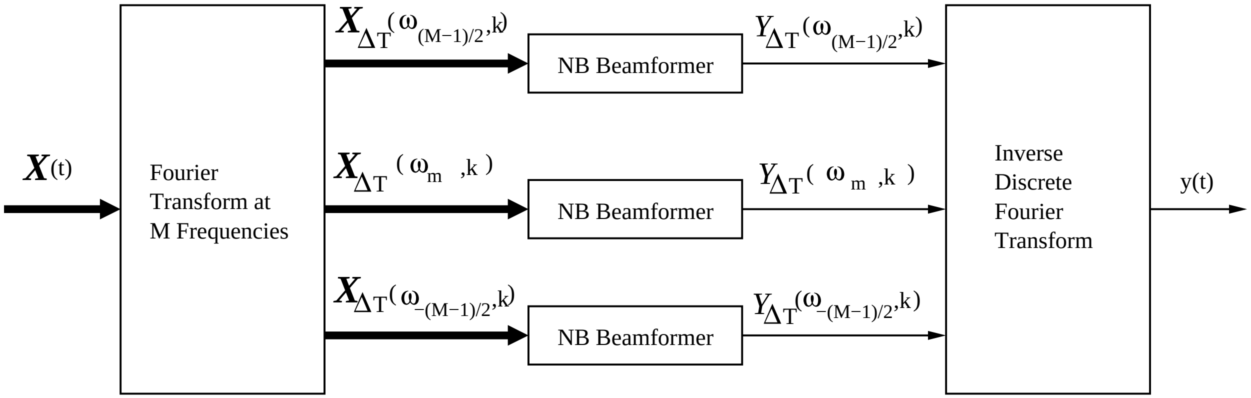
- In many applications, we implement a beamforming in the frequency domain ($\omega_m = \omega_c + m\frac{2\pi}{\Delta T}$ and M varies from $-\frac{M-1}{2}$ to $\frac{M-1}{2}$ if odd and from $-\frac{M}{2}$ to $\frac{M}{2} - 1$ if even).



$\boldsymbol{X}(t)$

Fourier Transform at M Frequencies

$\boldsymbol{X}_{\Delta T}(\omega_{(M-1)/2},k)$

$\boldsymbol{X}_{\Delta T}(\omega_m,k)$

$\boldsymbol{X}_{\Delta T}(\omega_{-(M-1)/2},k)$

NB Beamformer

NB Beamformer

NB Beamformer

$Y_{\Delta T}(\omega_{(M-1)/2},k)$

$Y_{\Delta T}(\omega_m,k)$

$Y_{\Delta T}(\omega_{-(M-1)/2},k)$

Inverse Discrete Fourier Transform

y(t)

MVDR Beamformer

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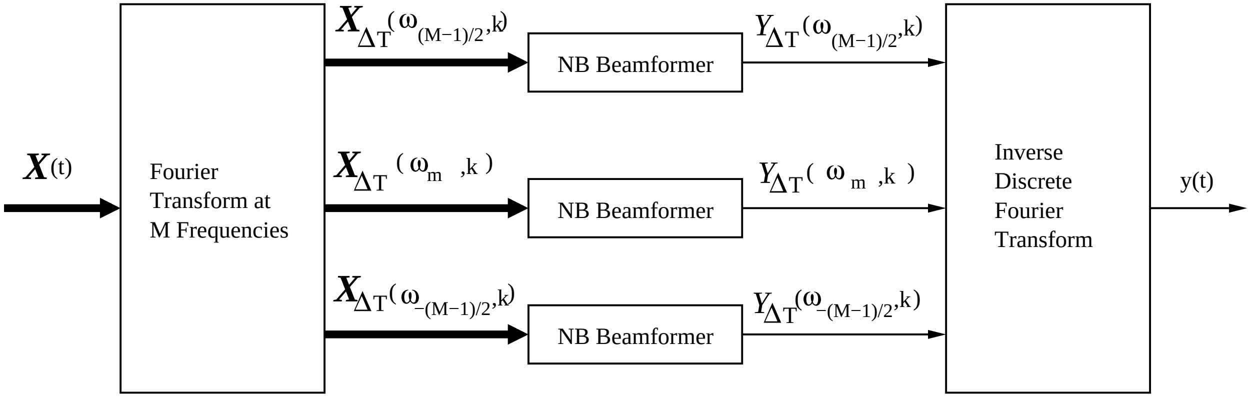
- In many applications, we implement a beamforming in the frequency domain ($\omega_m = \omega_c + m\frac{2\pi}{\Delta T}$ and M varies from $-\frac{M-1}{2}$ to $\frac{M-1}{2}$ if odd and from $-\frac{M}{2}$ to $\frac{M}{2} - 1$ if even).



$\boldsymbol{X}(t)$

Fourier Transform at M Frequencies

$\boldsymbol{X}_{\Delta T}(\omega_{(M-1)/2}, k)$

$\boldsymbol{X}_{\Delta T}(\omega_m, k)$

$\boldsymbol{X}_{\Delta T}(\omega_{-(M-1)/2}, k)$

NB Beamformer

NB Beamformer

NB Beamformer

$Y_{\Delta T}(\omega_{(M-1)/2}, k)$

$Y_{\Delta T}(\omega_m, k)$

$Y_{\Delta T}(\omega_{-(M-1)/2}, k)$

Inverse Discrete Fourier Transform

y(t)

- In order to generate these vectors, divide the observation interval T in K disjoint intervals of duration ΔT: $(k-1)\Delta T \leq t < k\Delta T, k = 1, \cdots, K$.

MVDR Beamformer

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- Assume an input signal with BW B_s centered in f_c
- In order to develop the frequency-domain snapshot model for the case in which the desired signals and the interfering signals can de modeled as plane waves, we have two cases: desired signals are deterministic or samples of a random process.

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- The signal vector can be written as $\boldsymbol{X}_s(\omega) = F(\omega)\boldsymbol{v}(\omega : \boldsymbol{k}_s)$ where $F(\omega)$ is the frequency-domain snapshot of the source signal and $\boldsymbol{v}(\omega : \boldsymbol{k}_s)$ is the array manifold vector for a plane-wave with wavenumber $\boldsymbol{k}_s$.

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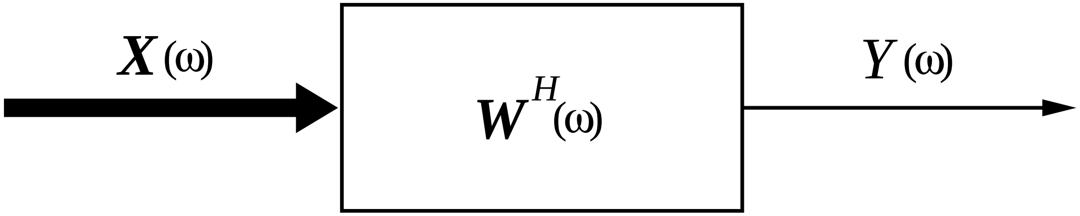
- The noise snapshot is a zero-mean random vector $\boldsymbol{N}(\omega)$ with spectral matrix given by $\boldsymbol{S}_{\boldsymbol{n}}(\omega) = \boldsymbol{S}_{\boldsymbol{c}}(\omega) + \sigma_\omega^2\boldsymbol{I}$

MVDR Beamformer

- We process $\boldsymbol{X}(\omega)$ with the $1 \times N$ operator $\boldsymbol{W}^H(\omega)$:



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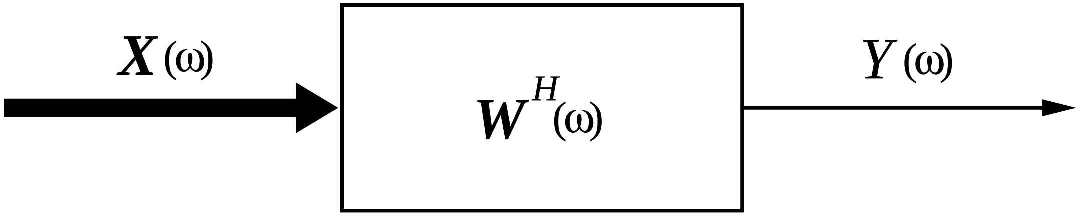

- Distortionless criterion (in the absence of noise):

$$
\begin{aligned}
Y(\omega) &= F(\omega) \\
&= \boldsymbol{W}^H(\omega)\boldsymbol{X}_s(\omega) = F(\omega)\boldsymbol{W}^H(\omega)\boldsymbol{v}(\omega : \boldsymbol{k}_s) \\
\Longrightarrow & \boldsymbol{W}^H(\omega)\boldsymbol{v}(\omega : \boldsymbol{k}_s) = 1
\end{aligned}
$$

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MVDR Beamformer

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- The mean square of the output noise is:

$$E[|Y_n(\omega)|^2] = \boldsymbol{W}^H(\omega)\boldsymbol{S}_n(\omega)\boldsymbol{W}(\omega)$$

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$$\boldsymbol{W}^H_{mvdr} = \Lambda_s \boldsymbol{v}^H \boldsymbol{S}_n^{-1} \text{ where } \Lambda_s = \left[\boldsymbol{v}^H \boldsymbol{S}_n^{-1} \boldsymbol{v}\right]^{-1}$$

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- This result is referred to as MVDR or *Capon* Beamformer.

Constrained Optimal Filtering

- The *gradient* of ξ with respect to $\boldsymbol{w}$ (real case):

$$\boldsymbol{\nabla}_{\boldsymbol{w}}\xi = \begin{bmatrix} \frac{\partial \xi}{\partial w_0} \\ \frac{\partial \xi}{\partial w_1} \\ \vdots \\ \frac{\partial \xi}{\partial w_{N-1}} \end{bmatrix}$$

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- which, when $\mathbf{R}$ is symmetric, leads to $\boldsymbol{\nabla}_{\boldsymbol{w}}(\boldsymbol{w}^T\boldsymbol{R}\boldsymbol{w}) = 2\boldsymbol{R}\boldsymbol{w}$

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- $\cdots$ which corresponds to $\nabla_{\boldsymbol{w}}\xi = \nabla_{\boldsymbol{a}}\xi + j\nabla_{\boldsymbol{b}}\xi$
- Let us define the *derivative* $\frac{\partial}{\partial\mathbf{w}}$ (with respect to $\mathbf{w}$):

$$\frac{\partial}{\partial\mathbf{w}} = \frac{1}{2}\begin{bmatrix} \frac{\partial}{\partial a_0} - j\frac{\partial}{\partial b_0} \\ \frac{\partial}{\partial a_1} - j\frac{\partial}{\partial b_1} \\ \vdots \\ \frac{\partial}{\partial a_{N-1}} - j\frac{\partial}{\partial b_{N-1}} \end{bmatrix}$$

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- The *conjugate derivative* with respect to $\mathbf{w}$ is

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- $\nabla_{\mathbf{w}} E[e(k)e^*(k)] = E\{e^*(k)[\nabla_{\mathbf{w}} e(k)] + e(k)[\nabla_{\mathbf{w}} e^*(k)]\}$

Constrained Optimal Filtering

- We compute each gradient ...

$$\begin{aligned}\boldsymbol{\nabla}_{\mathbf{w}} e(k) =& \boldsymbol{\nabla}_{\mathbf{a}}[d(k) - \mathbf{w}^H \mathbf{x}(k)] + j\boldsymbol{\nabla}_{\mathbf{b}}[d(k) - \mathbf{w}^H \mathbf{x}(k)] \\ =& - \mathbf{x}(k) - \mathbf{x}(k) = -2\mathbf{x}(k)\end{aligned}$$

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- such that the final result is

$$\begin{aligned}\nabla_{\mathbf{w}} E[e(k)e^*(k)] =& -2E[e^*(k)\mathbf{x}(k)] \\ =& -2E[\mathbf{x}(k)[d(k) - \mathbf{w}^H \mathbf{x}(k)]^*\} \\ =& -2\underbrace{E[\mathbf{x}(k)d^*(k)]}_{\mathbf{p}} + 2\underbrace{E[\mathbf{x}(k)\mathbf{x}^H(k)]}_{\mathbf{R}}\mathbf{w}\end{aligned}$$

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- The output of the processor is $y(k) = \mathbf{w}^H\mathbf{x}(k)$.
- It is worth mentioning that the most general case corresponds to having a reference signal, $d(k)$. It is, however, usual to have no reference signal as in Linearly-Constrained Minimum-Variance (LCMV) applications. In LCMV, if $\mathbf{f} = 1$, the system is often referred to as Minimum-Variance Distortionless Response (MVDR).

Constrained Optimal Filtering

- Using Lagrange multipliers, we form

$$\xi(k) = E[e(k)e^*(k)] + \boldsymbol{\mathcal{L}}_R^T Re[\mathbf{C}^H\mathbf{w} - \mathbf{f}] + \boldsymbol{\mathcal{L}}_I^T Im[\mathbf{C}^H\mathbf{w} - \mathbf{f}]$$

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- We can also represent the above expression with a complex $\boldsymbol{\mathcal{L}}$ given by $\boldsymbol{\mathcal{L}}_R + j\boldsymbol{\mathcal{L}}_I$ such that

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- Noting that $e(k) = d(k) - \mathbf{w}^H\mathbf{x}(k)$, we compute:

$$\begin{aligned}\boldsymbol{\nabla}_{\mathbf{w}}\xi(k) &= \boldsymbol{\nabla}_{\mathbf{w}}\left\{E[e(k)e^*(k)] + \frac{1}{2}\boldsymbol{\mathcal{L}}^H(\mathbf{C}^H\mathbf{w} - \mathbf{f}) + \frac{1}{2}\boldsymbol{\mathcal{L}}^T(\mathbf{C}^T\mathbf{w}^* - \mathbf{f}^*)\right\} \\ &= E[-2\mathbf{x}(k)e^*(k)] + \mathbf{0} + \mathbf{C}\boldsymbol{\mathcal{L}} \\ &= -2E[\mathbf{x}(k)d^*(k)] + 2E[\mathbf{x}(k)\mathbf{x}^H(k)]\mathbf{w} + \mathbf{C}\boldsymbol{\mathcal{L}}\end{aligned}$$

Constrained Optimal Filtering

- By using $\mathbf{R} = E[\mathbf{x}(k)\mathbf{x}^H(k)]$ and $\mathbf{p} = E[d^*(k)\mathbf{x}(k)]$, the gradient is equated to zero and the results can be written as (note that stationarity was assumed for the input and reference signals): $-2\mathbf{p} + 2\mathbf{R}\mathbf{w} + \mathbf{C}\boldsymbol{\mathcal{L}} = \mathbf{0}$

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- If we pre-multiply the previous expression by $\mathbf{C}^H$ and use $\mathbf{C}^H\mathbf{w} = \mathbf{f}$, we find $\boldsymbol{\mathcal{L}}$:
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- By replacing $\boldsymbol{\mathcal{L}}$, we obtain the *Wiener solution* for the linearly constrained adaptive filter:
$\mathbf{w}_{opt} = \mathbf{R}^{-1}\mathbf{p} + \mathbf{R}^{-1}\mathbf{C}(\mathbf{C}^H\mathbf{R}^{-1}\mathbf{C})^{-1}(\mathbf{f} - \mathbf{C}^H\mathbf{R}^{-1}\mathbf{p})$

Constrained Optimal Filtering

- The optimal solution for LCAF:
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- Also note that in case we do not have constraints ($\mathbf{C}$ and $\mathbf{f}$ are nulls), the optimal solution above becomes the *unconstrained* Wiener solution $\mathbf{R}^{-1}\mathbf{p}$.

The GSC

We start by doing a transformation in the coefficient vector.

- Let $\mathbf{T} = [\mathbf{C}\ \mathbf{B}]$ such that

$$\mathbf{w} = \mathbf{T}\bar{\mathbf{w}} = [\mathbf{C}\ \mathbf{B}] \begin{bmatrix} \bar{\mathbf{w}}_U \\ -\bar{\mathbf{w}}_L \end{bmatrix} = \mathbf{C}\bar{\mathbf{w}}_U - \mathbf{B}\bar{\mathbf{w}}_L$$

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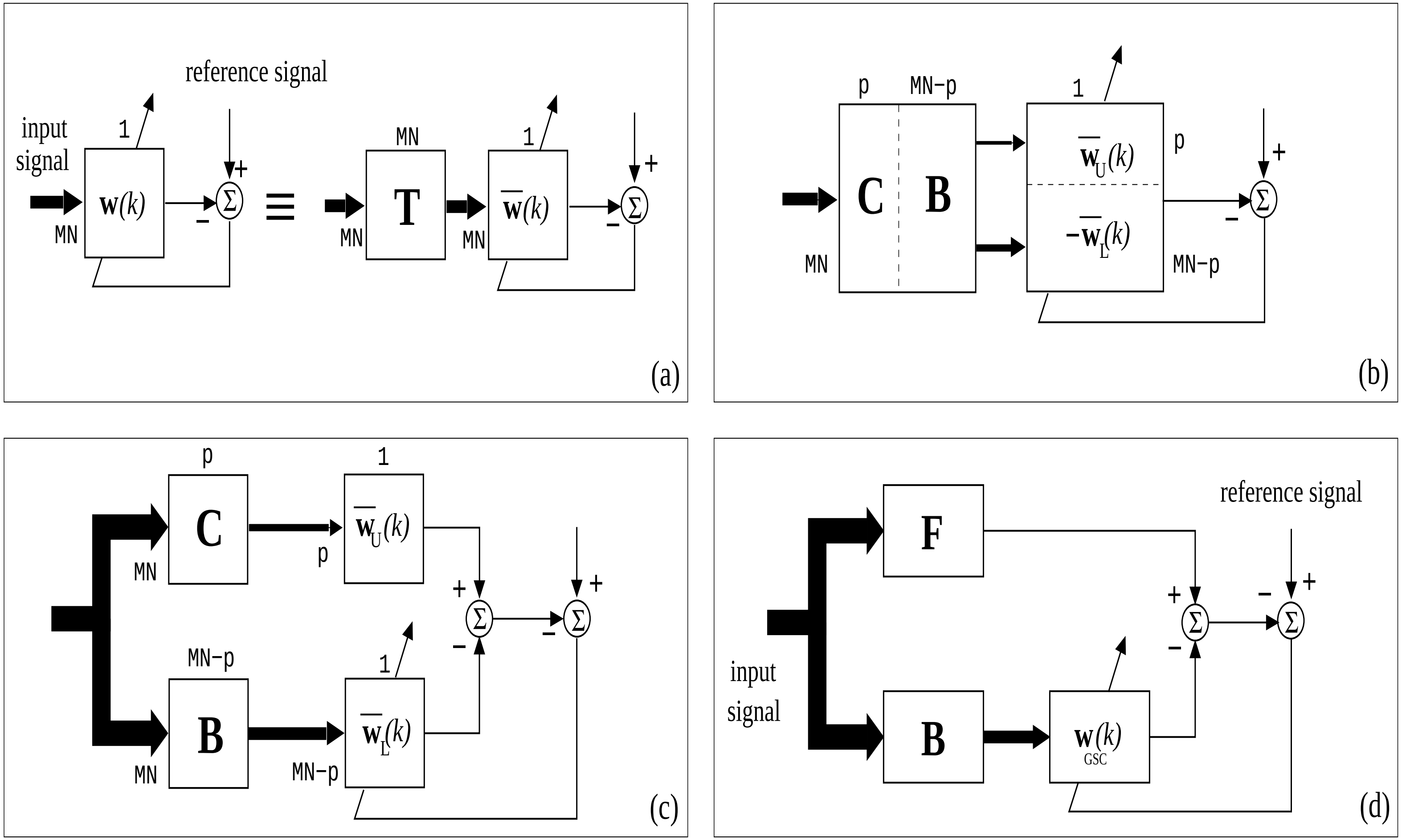
- $\bar{\mathbf{w}}_U$ is fixed and termed the quiescent weight vector; the minimization process will be carried out only in the lower part, also designated $\mathbf{w}_{GSC} = \bar{\mathbf{w}}_L$.

The GSC

- It is shown below how to split the transformation matrix into two parts: a fixed path and an *adaptive* path.



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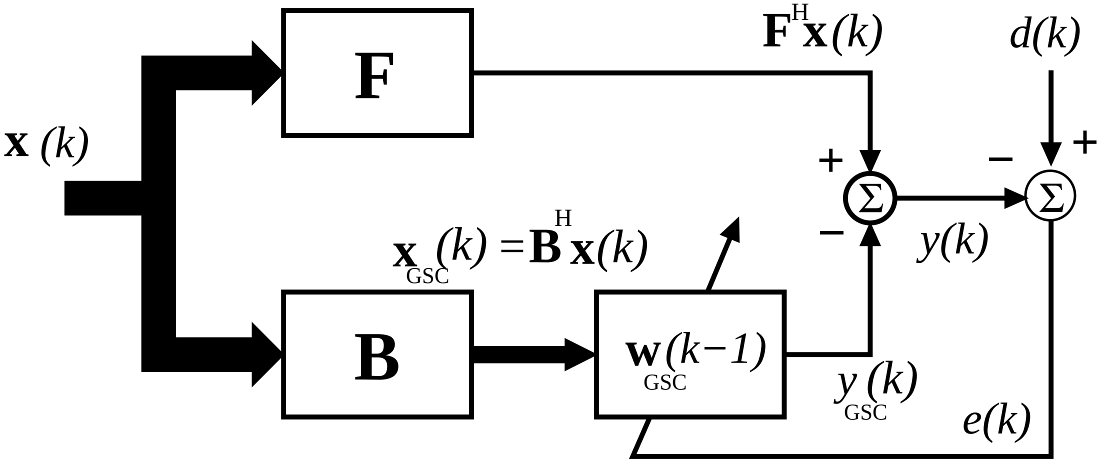
- This structure (detailed below) was named the Generalized Sidelobe Canceller (GSC).



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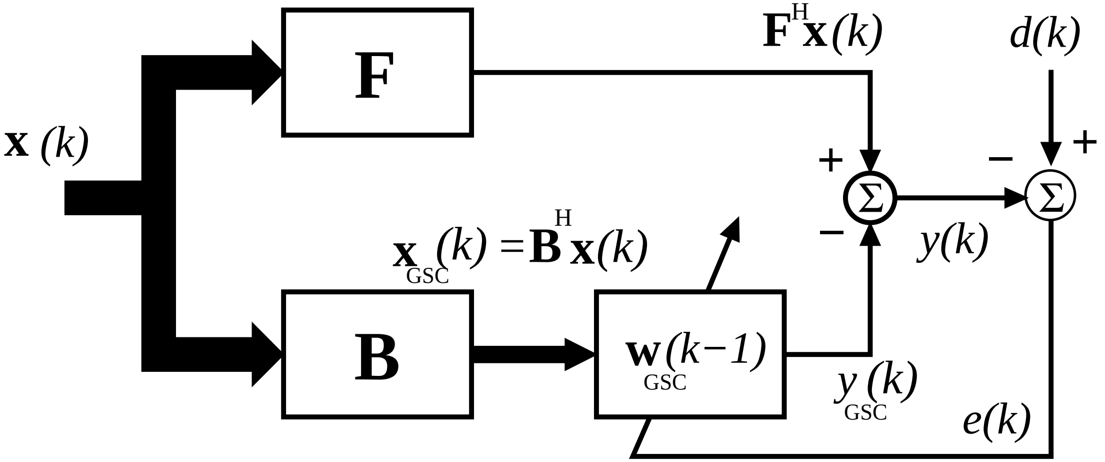

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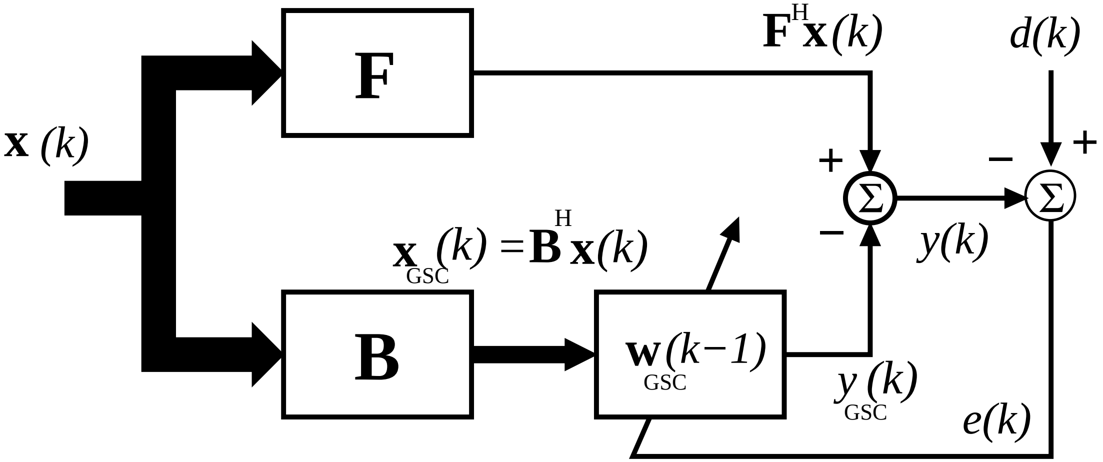

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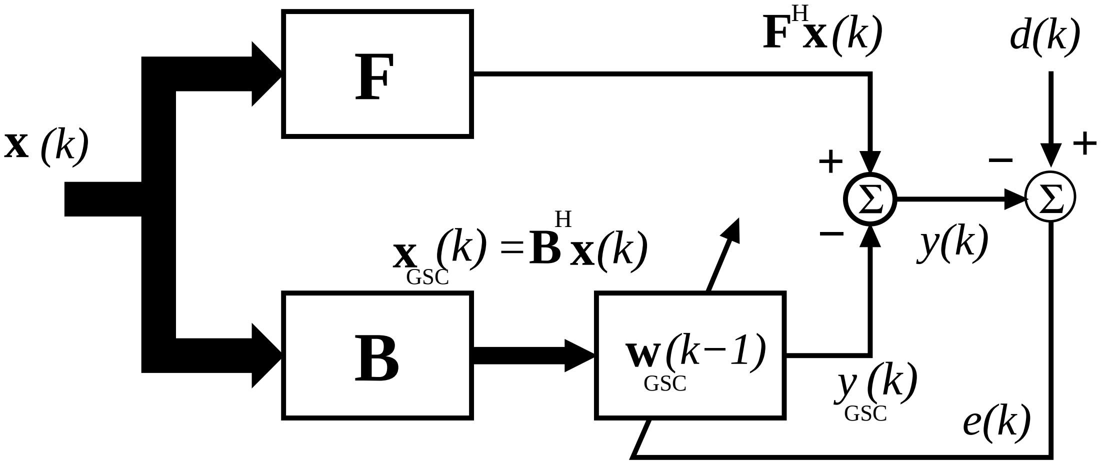

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- If we pre-multiply last equation by B^H and isolate $\mathbf{w}_{GSC}$, we find $\mathbf{w}_{GSC} = -(\mathbf{B}^H\mathbf{B})^{-1}\mathbf{B}^H\mathbf{w}$.
- Knowing that $\mathbf{T} = [\mathrm{C}\ \mathrm{B}]$ and that $\mathbf{T}^H\mathbf{T} = \boldsymbol{I}$, it follows that $\boldsymbol{P} = \mathbf{I} - \mathrm{C}(\mathrm{C}^H\mathrm{C})^{-1}\mathrm{C}^H = \mathrm{B}(\mathrm{B}^H\mathrm{B})\mathrm{B}^H$.

The GSC

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- The cross-correlation vector is given as:

$$\begin{aligned}\mathbf{p}_{GSC} &= E[d_{GSC}^{*}\mathbf{x}_{GSC}] \\ &= E\{[\mathbf{F}^{H}\mathbf{x} - d]^{*}[\mathbf{B}^{H}\mathbf{x}]\} \\ &= E[-\mathbf{B}^{H}d^{*}\mathbf{x} + \mathbf{B}^{H}\mathbf{x}\mathbf{x}^{H}\mathbf{F}] \\ &= -\mathbf{B}^{H}\mathbf{p} + \mathbf{B}^{H}\mathbf{R}\mathbf{F}\end{aligned}$$

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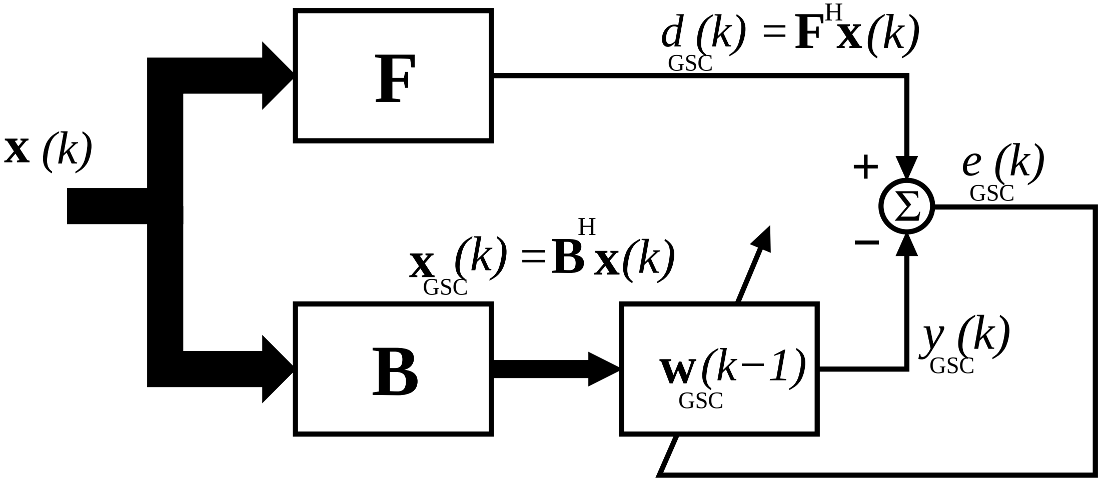
- $\cdots$ and $\mathbf{w}_{GSC-OPT} = (\mathbf{B}^H\mathbf{R}\mathbf{B})^{-1}(-\mathbf{B}^H\mathbf{p} + \mathbf{B}^H\mathbf{R}\mathbf{F})$

- A common case is when $d(k) = 0$:



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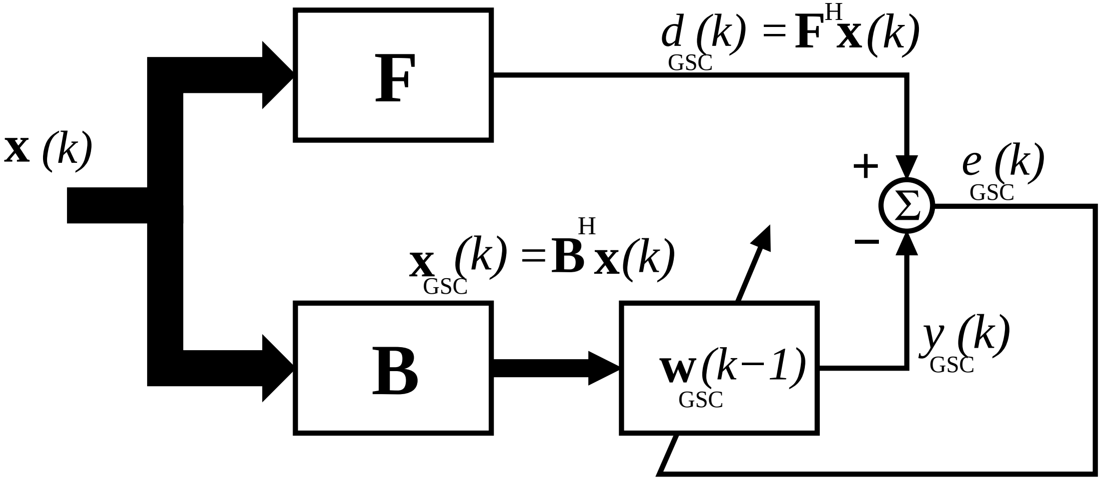

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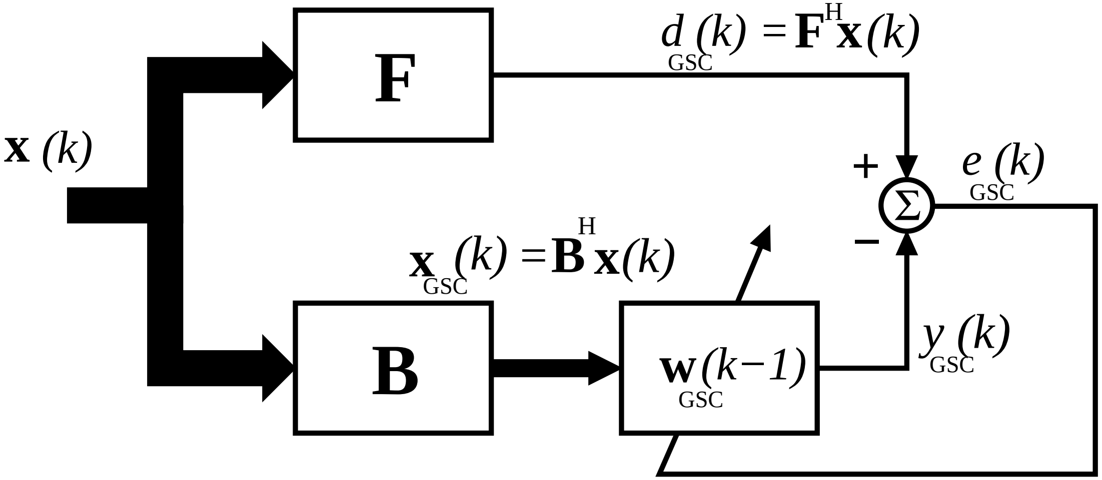

- We have dropped the negative sign that should exist according to the notation used. Although we define $e_{GSC}(k) = -e(k)$, the inversion of the sign in the error signal actually results in the same results because the error function is always based on the absolute value.
- In this case, the optimum filter $\mathbf{w}_{OPT}$ is: $\mathbf{F} - \mathbf{B}\mathbf{w}_{GSC-OPT} = \mathbf{F} - \mathbf{B}(\mathbf{B}^H\mathbf{R}\mathbf{B})^{-1}\mathbf{B}^H\mathbf{R}\mathbf{F} = \mathbf{R}^{-1}\mathbf{C}(\mathbf{C}^H\mathbf{R}^{-1}\mathbf{C})^{-1}\mathbf{f}$ (LCMV solution)

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- Let us recall the paper by Griffiths and Jim where the term GSC was coined; let

$$\mathbf{C}^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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$$\mathbf{B}_2^T =$$

$$\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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B_3^T is given by:

$$\begin{bmatrix}
-0.50 & -0.17 & -0.17 & 0.83 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.25 & -0.42 & 0.08 & 0.08 & 0.75 & -0.25 & -0.25 & -0.25 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.25 & -0.42 & 0.08 & 0.08 & -0.25 & 0.75 & -0.25 & -0.25 & 0.00 & 0.00 & 0.00 & 0.00 \\
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0.25 & 0.08 & -0.42 & 0.08 & 0.00 & 0.00 & 0.00 & 0.00 & -0.25 & -0.25 & 0.75 & -0.25 \\
0.25 & 0.08 & -0.42 & 0.08 & 0.00 & 0.00 & 0.00 & 0.00 & -0.25 & -0.25 & -0.25 & 0.75
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- Finally, a new efficient linearly constrained adaptive scheme which can also be visualized as a GSC structure can be found in [Campos&Werner&Apolinário IEEE-TSP Sept. 2002].

Outline

1. Introduction and Fundamentals
2. Sensor Arrays and Spatial Filtering
3. Optimal Beamforming
4. Adaptive Beamforming
5. DoA Estimation with Microphone Arrays

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4.1 Introduction

Introduction

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- Different algorithms may be employed for iteratively approximating the desired solution.
- We will briefly cover a small subset of algorithms for constrained adaptive filters.

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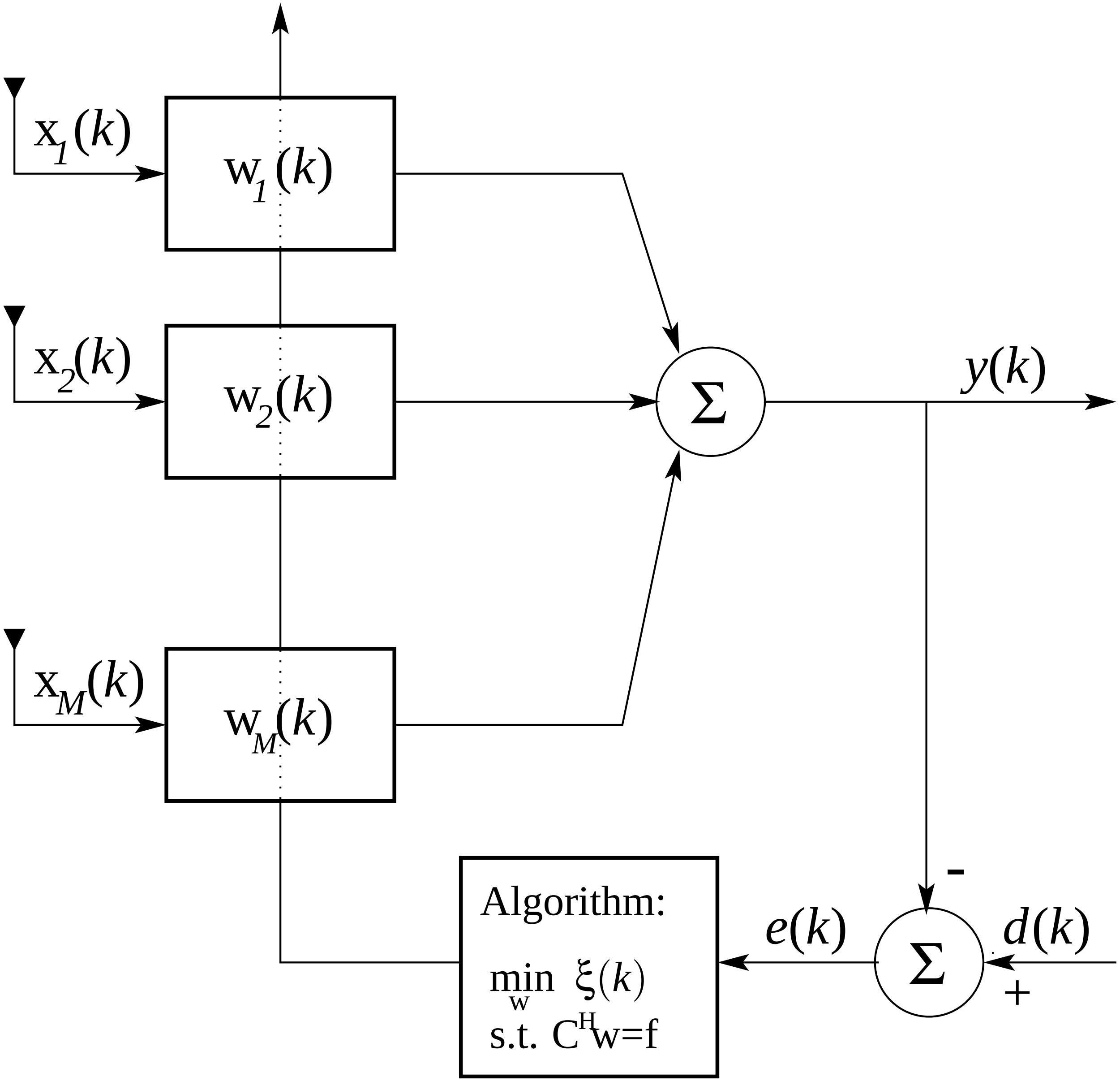
  - For example, if direction of arrival of the signal of interest is known, jammer suppression can take place through spatial filtering without the need of training signal, or in systems with constant-envelope modulation (e.g., M-PSK), a constant-modulus constraint can mitigate multipath propagation effects.

4.2 Constrained FIR Filters

Broadband Array Beamformer



Optimal Constrained MSE Filter

We look for

$$\min_{\mathbf{w}} \xi(k) \quad \text{s.t. } \mathbf{C}^H\mathbf{w} = \mathbf{f},$$

where

- $\xi(k) = E\left[|e(k)|^2\right]$
- $\mathbf{C}$ is the $MN \times p$ constraint matrix
- $\mathbf{f}$ is the $p \times 1$ gain vector

Optimal Constrained MSE Filter

The optimal beamformer is

$$\mathbf{w}(k) = \mathbf{R}^{-1}\mathbf{p} + \mathbf{R}^{-1}\mathbf{C}\left(\mathbf{C}^H\mathbf{R}^{-1}\mathbf{C}\right)^{-1}\left(\mathbf{f} - \mathbf{C}^H\mathbf{R}^{-1}\mathbf{p}\right)$$

where:

- $\mathbf{R} = E\left[\mathbf{x}(k)\mathbf{x}^H(k)\right]$ and $\mathbf{p} = E\left[d^*(k)\mathbf{x}(k)\right]$
- $\mathbf{w}(k) = \left[\mathbf{w}_1^T(k)\ \mathbf{w}_2^T(k)\ \cdots\ \mathbf{w}_M^T(k)\right]^T$
- $\mathbf{x}(k) = \left[\mathbf{x}_1^T(k)\ \mathbf{x}_2^T(k)\ \cdots\ \mathbf{x}_M^T(k)\right]^T$
- $\mathbf{x}_i^T(k) = [x_i(k)\ x_i(k-1)\ \cdots\ x_i(k-N+1)]$

The Constrained LS Beamformer

In the absence of statistical information, we may choose

$$\min_{\mathbf{w}} \left[\xi(k) = \sum_{i=0}^{k} \lambda^{k-i} |d(i) - \mathbf{w}^H \mathbf{x}(i)|^2 \right] \text{ s.t. } \mathbf{C}^H \mathbf{w} = \mathbf{f}$$

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with $\lambda \in (0, 1]$, which gives, as solution,

$$\begin{aligned} \mathbf{w}(k) &= \mathbf{R}^{-1}(k)\mathbf{p}(k) \\ &\quad + \mathbf{R}^{-1}(k)\mathbf{C}\left(\mathbf{C}^H\mathbf{R}^{-1}(k)\mathbf{C}\right)^{-1}\left[\mathbf{f} - \mathbf{C}^H\mathbf{R}^{-1}(k)\mathbf{p}(k)\right], \end{aligned}$$

where

$\mathbf{R}(k) = \sum_{i=0}^{k} \lambda^{k-i}\mathbf{x}(i)\mathbf{x}^H(i)$, and $\mathbf{p}(k) = \sum_{i=0}^{k} \lambda^{k-i} d^*(i)\mathbf{x}(i)$.

The Constrained LMS Algorithm

A (cheaper) alternative cost function is

$$\min_{\mathbf{w}} \left[\xi(k) = \|\mathbf{w}(k) - \mathbf{w}(k-1)\|^2 + \mu|e(k)|^2\right] \text{ s.t. } \mathbf{C}^H\mathbf{w}(k) = \mathbf{f},$$

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$$\mathbf{w}(k) = \mathbf{w}(k-1) + \mu e^*(k) \left[\mathbf{I} - \mathbf{C} \left(\mathbf{C}^H \mathbf{C} \right)^{-1} \mathbf{C}^H \right] \mathbf{x}(k),$$

where $e(k) = d(k) - \mathbf{w}^H(k-1)\mathbf{x}(k)$, μ is a positive small constant called step size.

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which gives, as solution,

$$\mathbf{w}(k) = \mathbf{P}\left[\mathbf{w}(k-1) + \mu e^*(k)\mathbf{x}(k)\right] + \mathbf{F},$$

where $e(k) = d(k) - \mathbf{w}^H(k-1)\mathbf{x}(k)$, μ is a positive small constant called step size, $\mathbf{P} = \mathbf{C}\left(\mathbf{C}^H\mathbf{C}\right)^{-1}\mathbf{C}^H$, and $\mathbf{F} = \mathbf{C}\left(\mathbf{C}^H\mathbf{C}\right)^{-1}\mathbf{f}$.

The Constrained AP Algorithm

We may wish to trade complexity for speed of convergence:

$$\min_{\mathbf{w}} \left[\xi(k) = \|\mathbf{w}(k) - \mathbf{w}(k-1)\|^2\right] \text{ s.t. } \begin{cases} \mathbf{X}^T(k)\mathbf{w}^*(k) = \mathbf{d}(k) \\ \mathbf{C}^H\mathbf{w}(k) = \mathbf{f}, \end{cases}$$

where

- $\mathbf{d}(k) = [d(k)\ d(k-1)\ \cdots\ d(k-L+1)]^T$
- $\mathbf{X}(k) = [\mathbf{x}(k)\ \mathbf{x}(k-1)\ \cdots\ \mathbf{x}(k-L+1)]^T$

The Constrained AP Algorithm

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which gives, as solution,

$$\mathbf{w}(k) = \mathbf{P}\left[\mathbf{w}(k-1) + \mu\mathbf{X}(k)\mathbf{t}(k)\right] + \mathbf{F}$$

where

- $\mathbf{e}(k) = \mathbf{d}(k) - \mathbf{X}^T(k)\mathbf{w}^*(k-1)$
- $\mathbf{t}(k) = \left[\mathbf{X}^H(k)\mathbf{P}\mathbf{X}(k)\right]^{-1}\mathbf{e}^*(k)$

Outline

1. Introduction and Fundamentals
2. Sensor Arrays and Spatial Filtering
3. Optimal Beamforming
4. Adaptive Beamforming
5. DoA Estimation with Microphone Arrays

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5.0 Signal Preparation

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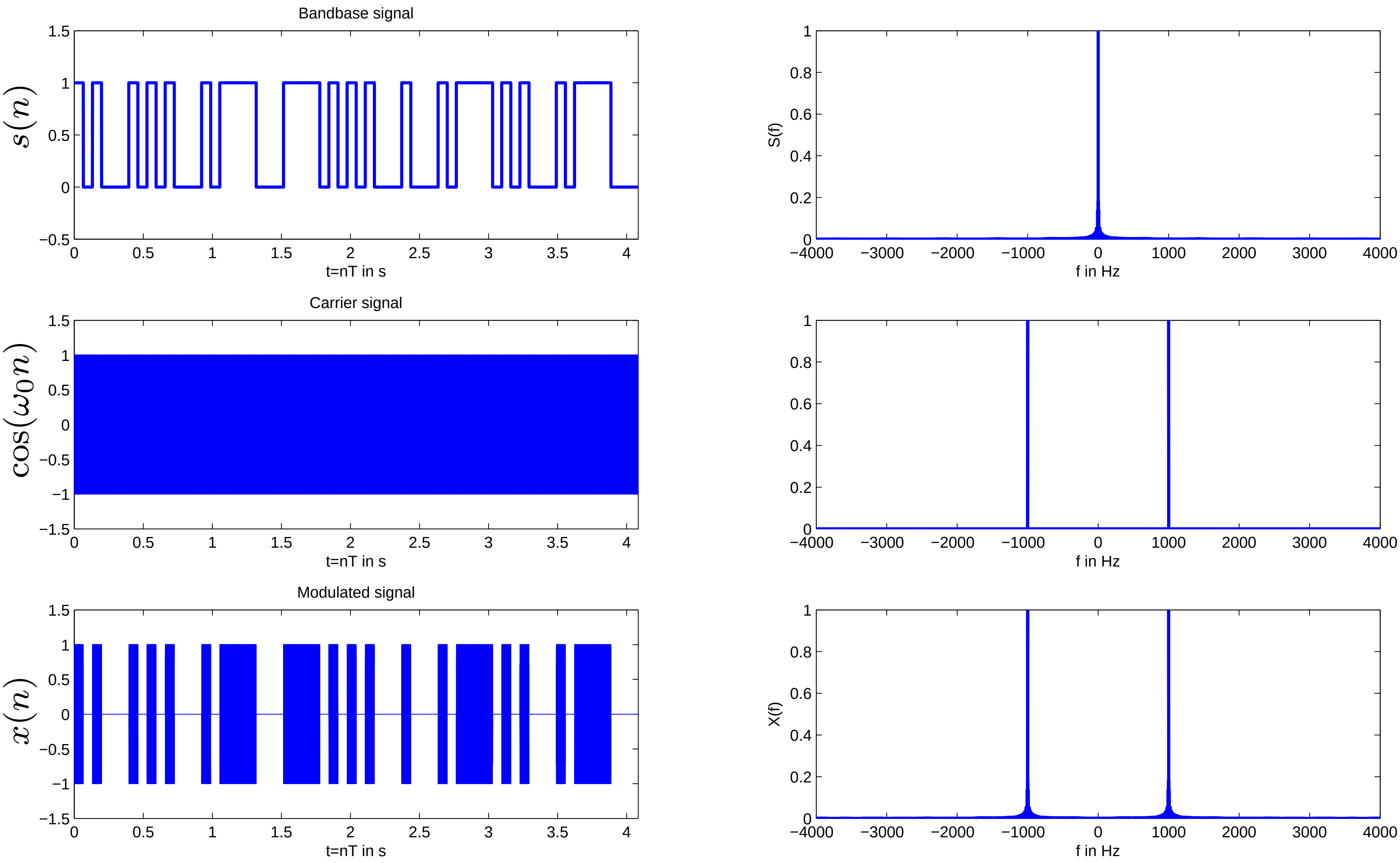
- But, most importantly, the delay is well represented only if the signal is also analytic, i. e., having only non-negative frequency components.

- An analytic signal, mathematically, can be obtained by multiplying its Fourier transform by the continuous Heaviside step function:

$$X_a(e^{j\omega}) = 2X(e^{j\omega})u(\omega), u(\omega) = \begin{cases} 0, \omega < 0 \\ 1, \omega = 0 \\ 1, \omega > 0 \end{cases}$$

- Let $x(n) = s(n)\cos(\omega_0 n)$, $s(n)$ having a maximum frequency component (ω_m) much lower than ω_0:



- If $x(n) = s(n)e^{j\omega_0 n}$, then
  $x(n)e^{-j\omega_0 \tau} = s(n)e^{j\omega_0 (n-\tau)} \approx x(n-\tau)$ if $\tau \ll 1/\omega_m$

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- But if $x(n) = s(n)cos(\omega_0 n)$, then $x(n)e^{-j\omega_0\tau} \neq x(n-\tau)$
- We can make $x(n) = s(n)cos(\omega_0 n) = \underbrace{\frac{s(n)}{2}e^{j\omega_0 n}}_{x_+(n)} + \underbrace{\frac{s(n)}{2}e^{-j\omega_0 n}}_{x_-(n)}$ such that $x(n-\tau) \approx x_+ e^{-j\omega_0\tau} + x_-(n)e^{+j\omega_0\tau} = s(n)cos(\omega_0(n-\tau))$

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- We can make $x(n) = s(n)cos(\omega_0 n) = \underbrace{\frac{s(n)}{2}e^{j\omega_0 n}}_{x_+(n)} + \underbrace{\frac{s(n)}{2}e^{-j\omega_0 n}}_{x_-(n)}$ such that
$$x(n-\tau) \approx x_+e^{-j\omega_0\tau} + x_-(n)e^{+j\omega_0\tau} = s(n)cos(\omega_0(n-\tau))$$
- $\cdots$ but, how to obtain $x_+(n)$ or a scaled copy? Using the Hilbert Transform $x_H(n) = \mathcal{HT}\{x(n)\}$ where
$$X_H(e^{j\omega}) = \begin{cases} jX(e^{j\omega}), -\pi < \omega < 0 \\ X(e^{j\omega}), \omega = 0 \\ -jX(e^{j\omega}), 0 < \omega < \pi \end{cases}$$

- Knowing that $x(n) = x_-(n) + x_+(n) = \mathcal{F}^{-1}\left\{X_-(e^{j\omega}) + X_+(e^{j\omega})\right\}$, we compute $y(n) = x(n) + jx_H(n)$

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$$= \mathcal{F}^{-1}\{X_-(e^{j\omega}) + X_+(e^{j\omega}) - X_-(e^{j\omega}) + X_+(e^{j\omega})\}$$

- Therefore $y(n) = 2\mathcal{F}^{-1}\{X_+(e^{j\omega})\} = s(n)e^{j\omega_0 n}$ which is analytic!

Signal Model

- Consider $x_m(t)$ the signal from the m-th microphone (prior to the A/D converter) corresponding to audio from D sources (directions θ_1 to θ_D) plus noise:
$x_m(t) = s_1(t - \bar{\tau}_m(\theta_1)) + \cdots + s_D(t - \bar{\tau}_m(\theta_D)) + n_m(t)$

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  $x_m(n) = s_1(n)e^{-j\omega_0\tau_m(\theta_1)} + \cdots + s_D(n)e^{-j\omega_0\tau_m(\theta_D)} + n_m(n)$

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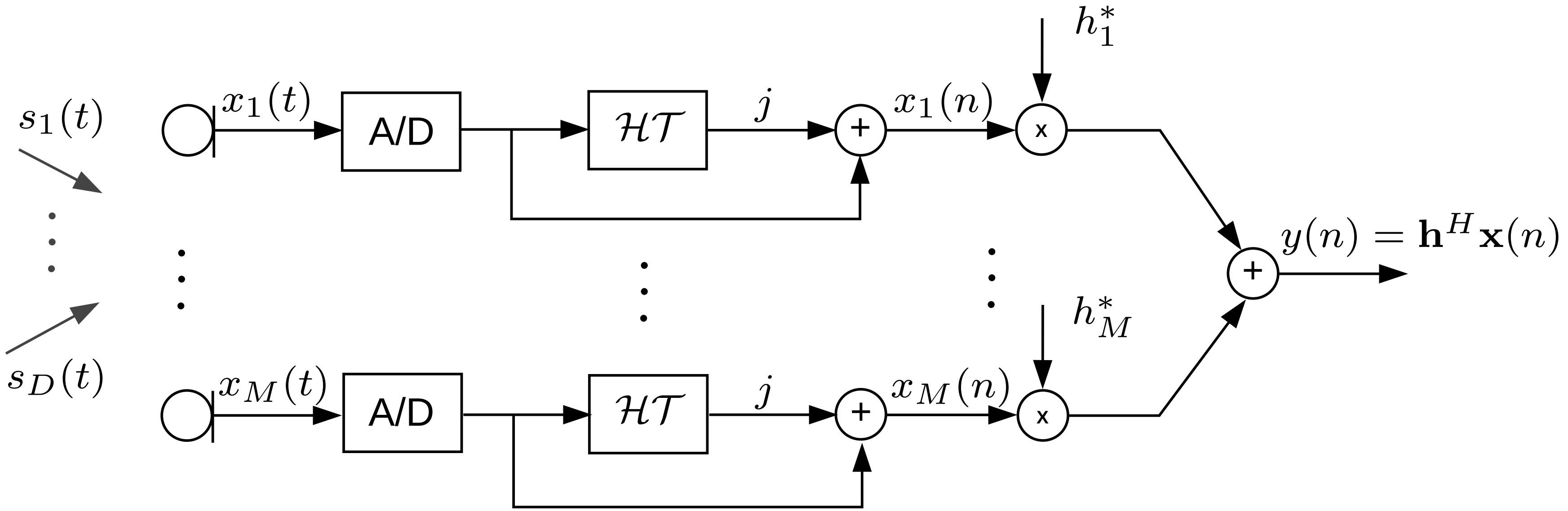
- For an array with M microphones, we would have:

$$\underbrace{\mathbf{x}(n)}_{M\times 1} = \underbrace{\mathbf{A}}_{M\times D}\underbrace{\mathbf{s}(n)}_{D\times 1} + \underbrace{\mathbf{n}(n)}_{M\times 1}$$

5.1 Signal model

- Assume, initially, we have D narrowband signals coming from unknown directions:



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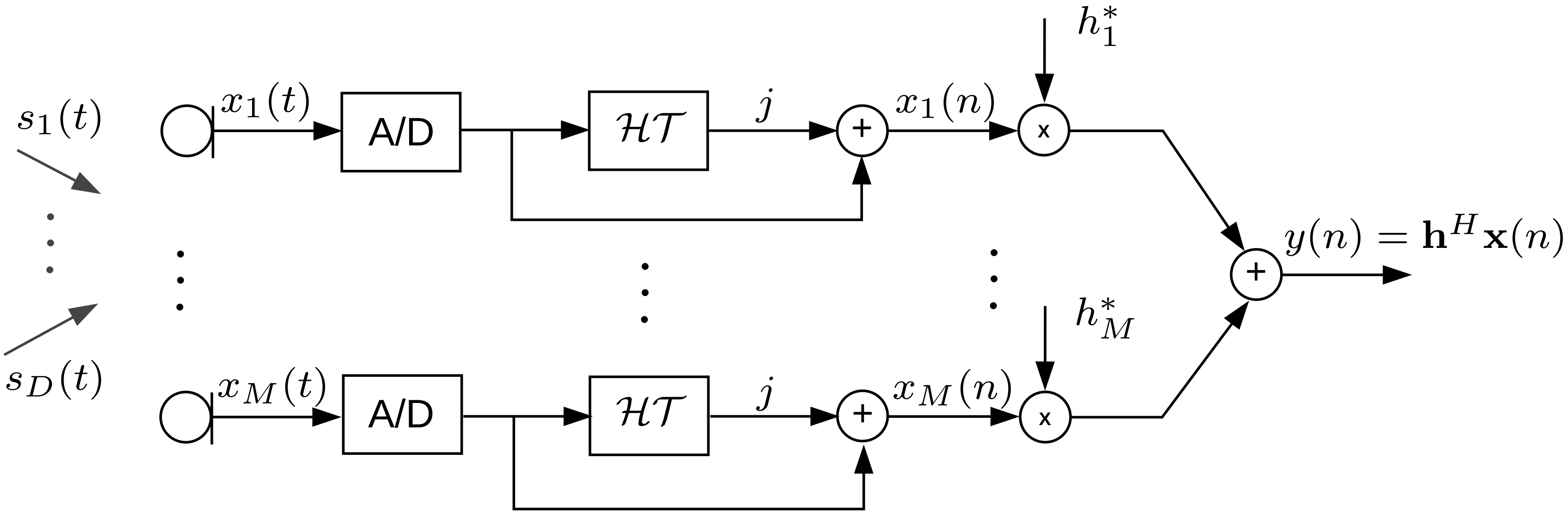

- $$\mathbf{x}(n) = \begin{bmatrix} e^{-j\omega_0\tau_1(\theta_1)}s_1(n) + \cdots + e^{-j\omega_0\tau_1(\theta_D)}s_D(n) + n_1(n) \\ \vdots \\ e^{-j\omega_0\tau_M(\theta_1)}s_1(n) + \cdots + e^{-j\omega_0\tau_M(\theta_D)}s_D(n) + n_M(n) \end{bmatrix}$$

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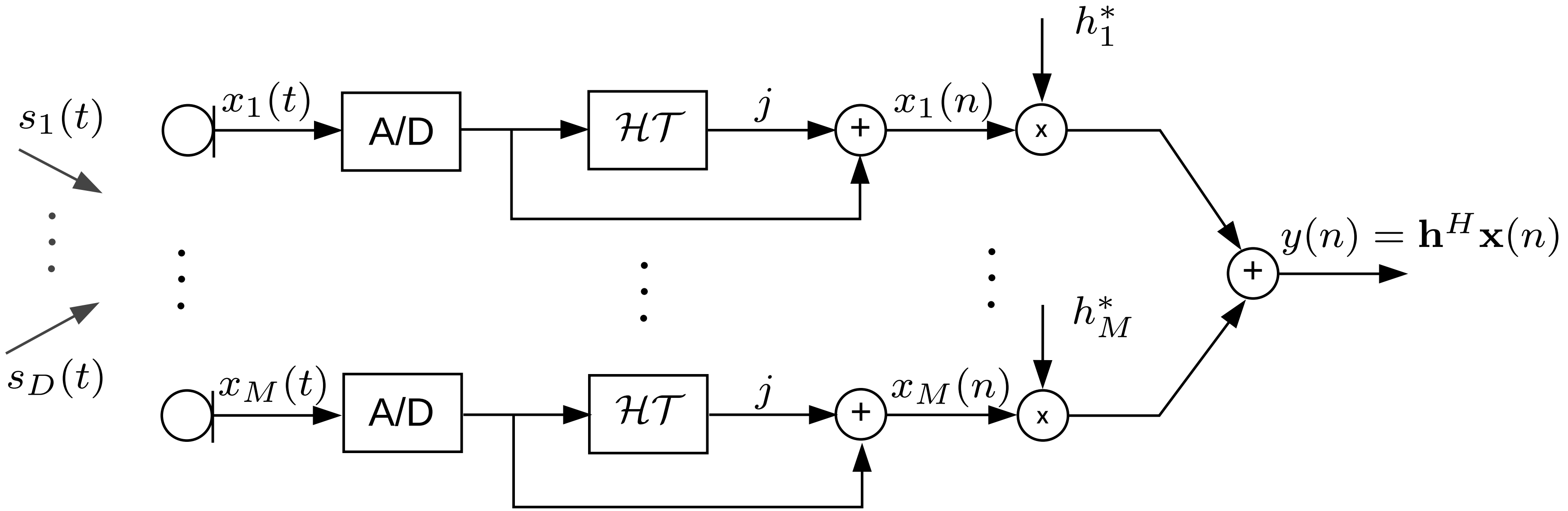

- $$\mathbf{x}(n) = \begin{bmatrix} e^{-j\omega_0\tau_1(\theta_1)}s_1(n) + \cdots + e^{-j\omega_0\tau_1(\theta_D)}s_D(n) + n_1(n) \\ \vdots \\ e^{-j\omega_0\tau_M(\theta_1)}s_1(n) + \cdots + e^{-j\omega_0\tau_M(\theta_D)}s_D(n) + n_M(n) \end{bmatrix}$$

- Such that the output signal can be written as
$y(n) = \mathbf{h}^H\mathbf{x}(n) = \mathbf{h}^H\left[\mathbf{A}\mathbf{s}(n) + \mathbf{n}(n)\right]$

- If we now assume one single signal, $s(n)$, coming from direction θ, then
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- Also note that $E[|y(n)|^2] = \mathbf{h}^H\mathbf{R}_x\mathbf{h}$, $\mathbf{R}_x = E[\mathbf{x}(n)\mathbf{x}^H(n)]$

5.2 Non-parametric methods: BF (beamforming a.k.a. Delay & Sum) and Capon

DS DoA

- If $\mathbf{x}(n)$ were spatially white, i.e. $\mathbf{R}_x = \mathbf{I}$, we would obtain $E[|y(n)|^2] = \mathbf{h}^H\mathbf{h}$

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  and equating to zero, is $\mathbf{h} = \mathbf{a}(\theta)/M$ which leads to $E[|y(n)|^2] = \frac{\mathbf{a}^H(\theta)\mathbf{R}_x\mathbf{a}(\theta)}{M^2}$

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- Omitting factor $\frac{1}{M^2}$, we estimate the autocorrelation matrix as $\hat{\mathbf{R}}_x = \frac{1}{N}\sum_{n=1}^{N}\mathbf{x}(n)\mathbf{x}^H(n)$ and find the direction of interest by varying θ and obtaining the peak in $\boxed{P_{DS}(\theta) = \mathbf{a}^H(\theta)\hat{\mathbf{R}}_x\mathbf{a}(\theta)}$

Capon

- In the method known as Capon, we minimize $E[|y(n)|^2] = \mathbf{h}^H \mathbf{R}_x \mathbf{h}$ subject to $\mathbf{h}^H \mathbf{a}(\theta) = 1$

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- Replacing the above coefficient vector in $E[|y(n)|^2]$, we obtain $E[|y(n)|^2] = \frac{1}{\mathbf{a}^H(\theta) \mathbf{R}_x^{-1} \mathbf{a}(\theta)}$

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- Replacing the above coefficient vector in $E[|y(n)|^2]$, we obtain $E[|y(n)|^2] = \frac{1}{\mathbf{a}^H(\theta) \mathbf{R}_x^{-1} \mathbf{a}(\theta)}$

- Therefore, in the Capon DoA, we estimate $\hat{\mathbf{R}}_x = \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}(n) \mathbf{x}^H(n)$ and find the direction of interest by varying θ and obtaining the peak in

$$\boxed{P_{CAPON}(\theta) = \frac{1}{\mathbf{a}^H(\theta) \hat{\mathbf{R}}_x^{-1} \mathbf{a}(\theta)}}$$

5.3 Eigenvalue-Based DoA

MUSIC

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- Also note that $\mathbf{A}$ is $M \times D$, $\mathbf{s}$ is $D \times 1$, and $\mathbf{n}(n)$ is $M \times 1$
- We then write $\mathbf{R}_x = E\left[\mathbf{x}(n)\mathbf{x}^H(n)\right] = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{R}_n$, this last matrix becoming $\mathbf{R}_n = \sigma_n^2\mathbf{I}$ when assuming spatially white noise; $\mathbf{R}_s$ is the $D \times D$ autocorrelation matrix of the signal vector, i.e., $E\left[\mathbf{s}(n)\mathbf{s}^H(n)\right]$

MUSIC

- $\mathbf{R}_x = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{R}_n$ with $D < M$ implies that $\mathbf{A}\mathbf{R}_s\mathbf{A}^H$ is singular (rank D), its determinant is equal to zero and, therefore, $\det\left[\mathbf{R}_x - \sigma_n^2\mathbf{I}\right] = 0$ and σ_n^2 is a (minimum) eigenvalue with multiplicity $M - D$

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- Spectral decomposition of matrix $\mathbf{R}_x$: vector $\mathbf{e}_m$ being an eigenvector of $\mathbf{R}_x$ means that $\mathbf{R}_x\mathbf{e}_m = \lambda_m\mathbf{e}_m$. Collecting all eigenvectors in matrix $\mathbf{E}$, we may write $\mathbf{R}_x\mathbf{E} = \mathbf{E}\boldsymbol{\Lambda} = \left[\mathbf{e}_1 \cdots \mathbf{e}_M\right]$ diag $\left\{\left[\lambda_1 \cdots \lambda_M\right]\right\}$
$\Rightarrow \mathbf{R}_x = \mathbf{E}\boldsymbol{\Lambda}\mathbf{E}^H$

MUSIC

- $\mathbf{R}_x = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{R}_n$ with $D < M$ implies that $\mathbf{A}\mathbf{R}_s\mathbf{A}^H$ is singular (rank D), its determinant is equal to zero and, therefore, $\det\left[\mathbf{R}_x - \sigma_n^2\mathbf{I}\right] = 0$ and σ_n^2 is a (minimum) eigenvalue with multiplicity $M - D$

- Spectral decomposition of matrix $\mathbf{R}_x$: vector $\mathbf{e}_m$ being an eigenvector of $\mathbf{R}_x$ means that $\mathbf{R}_x\mathbf{e}_m = \lambda_m\mathbf{e}_m$. Collecting all eigenvectors in matrix $\mathbf{E}$, we may write $\mathbf{R}_x\mathbf{E} = \mathbf{E}\mathbf{\Lambda} = \left[\mathbf{e}_1 \cdots \mathbf{e}_M\right]$ diag $\{[\lambda_1 \cdots \lambda_M]\}$
$\Rightarrow \mathbf{R}_x = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^H$

- Dividing matrix $\mathbf{E}$ in two parts, the first D columns and the last $N = M - D$ columns, we have:
$\mathbf{E} = [\underbrace{\mathbf{e}_1 \cdots \mathbf{e}_D}_{\mathbf{E}_S}\,\underbrace{\mathbf{e}_{D+1} \cdots \mathbf{e}_M}_{\mathbf{E}_N}] = [\mathbf{E}_S \;\; \mathbf{E}_N]$

MUSIC

- Noting that $\mathbf{E}\mathbf{E}^H = \mathbf{I}$, we can write $\mathbf{E}_S\mathbf{E}_S^H + \mathbf{E}_N\mathbf{E}_N^H = \mathbf{I}$

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- We can find the distance d from a vector $\mathbf{v}$ to the signal subspace $\mathbf{E}_S$ by obtaining $\mathbf{x}$ that minimizes $d = |\mathbf{v} - \mathbf{E}_S\mathbf{x}|$; the result is $d^2 = \mathbf{v}^H\mathbf{E}_N\mathbf{E}_N^H\mathbf{v}$

MUSIC

- The squared distance from vector $\mathbf{a}(\theta)$ to the signal subspace (spanned by $\mathbf{E}_S$) is $d^2 = \mathbf{a}^H(\theta)\mathbf{E}_N\mathbf{E}_N^H\mathbf{a}(\theta)$

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- Its inverse will present peaks. In algorithm MUSIC, we estimate D from the eigenvalues of $\hat{\mathbf{R}}_x$; from its eigenvectors, we form $\mathbf{E}_S$ and $\mathbf{E}_N$, and by varying θ, we shall find peaks in the directions of θ_1 to θ_D in

$$\boxed{P_{MUSIC}(\theta) = \frac{1}{d^2_{\mathbf{a}(\theta)}} = \frac{1}{\mathbf{a}^H(\theta)\mathbf{E}_N\mathbf{E}_N^H\mathbf{a}(\theta)}}$$

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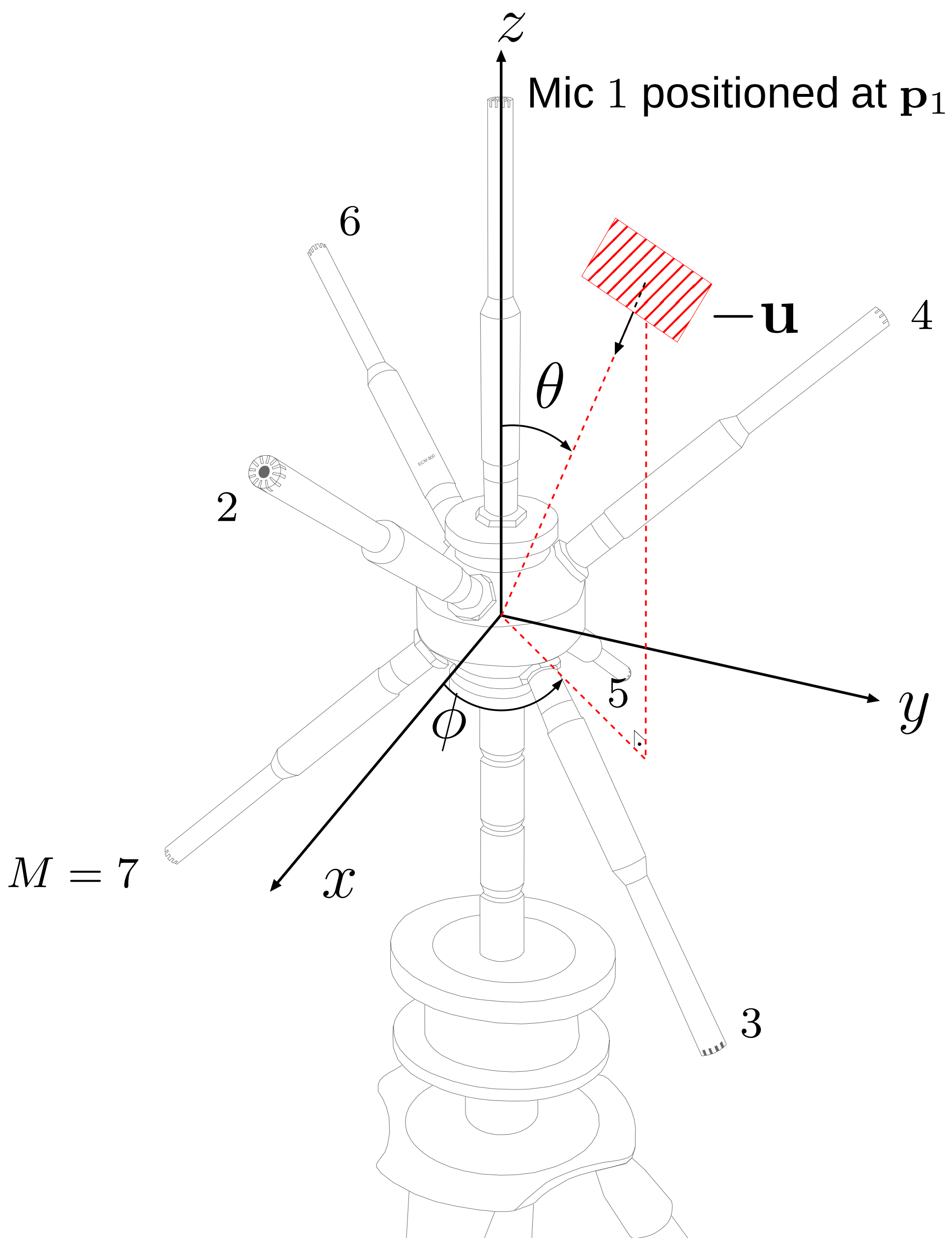
- If $\mathbf{R}_S$ is required, we compute

$$\mathbf{R}_S = \left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H\left(\mathbf{R}_x - \sigma_n^2\mathbf{I}\right)\mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}$$

5.4 GCC-Based DoA

- M microphones of an array are in positions $\mathbf{p}_1$ to $\mathbf{p}_M$:



GCC

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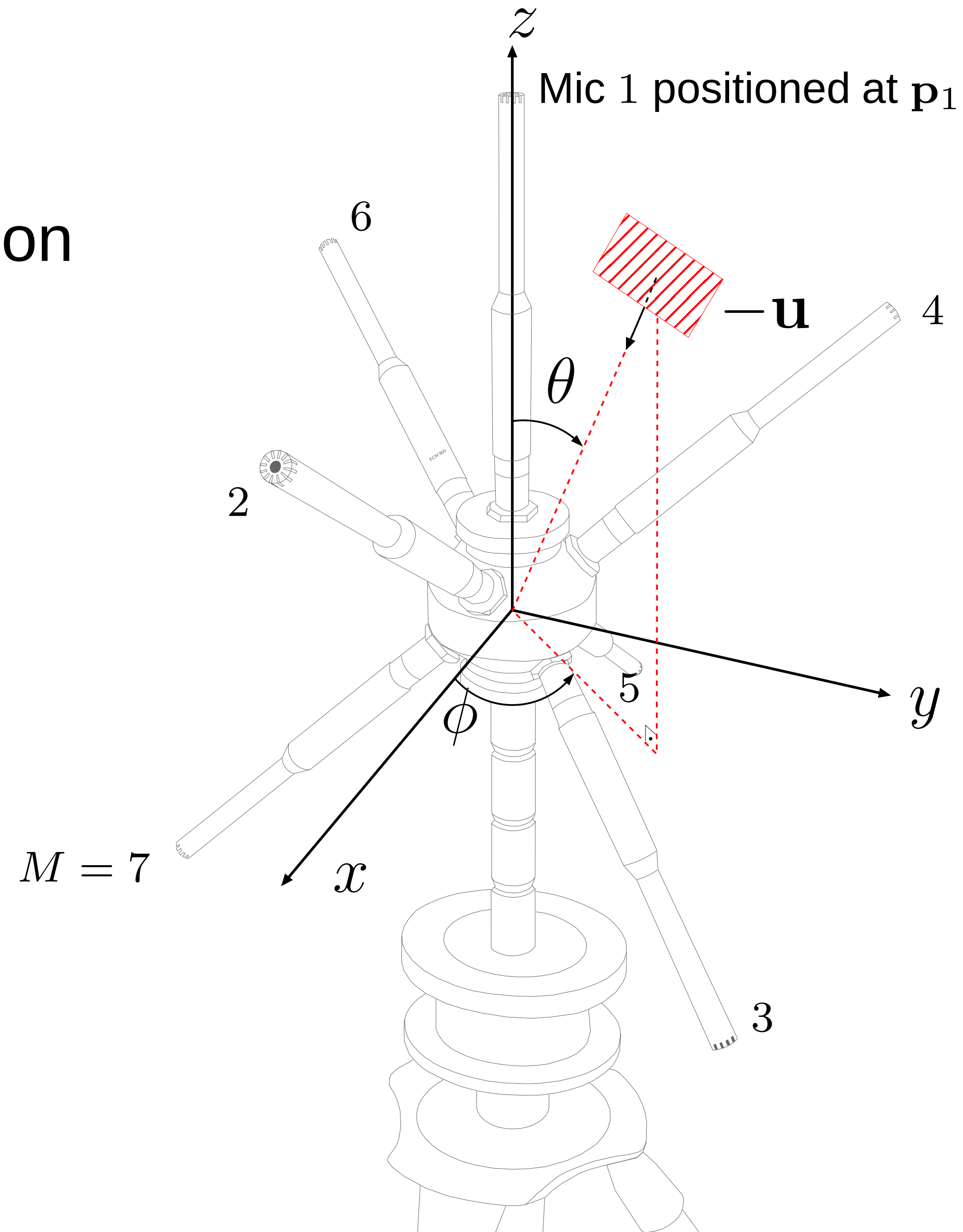
- $-\mathbf{u}$: unit vector in the direction of propagation



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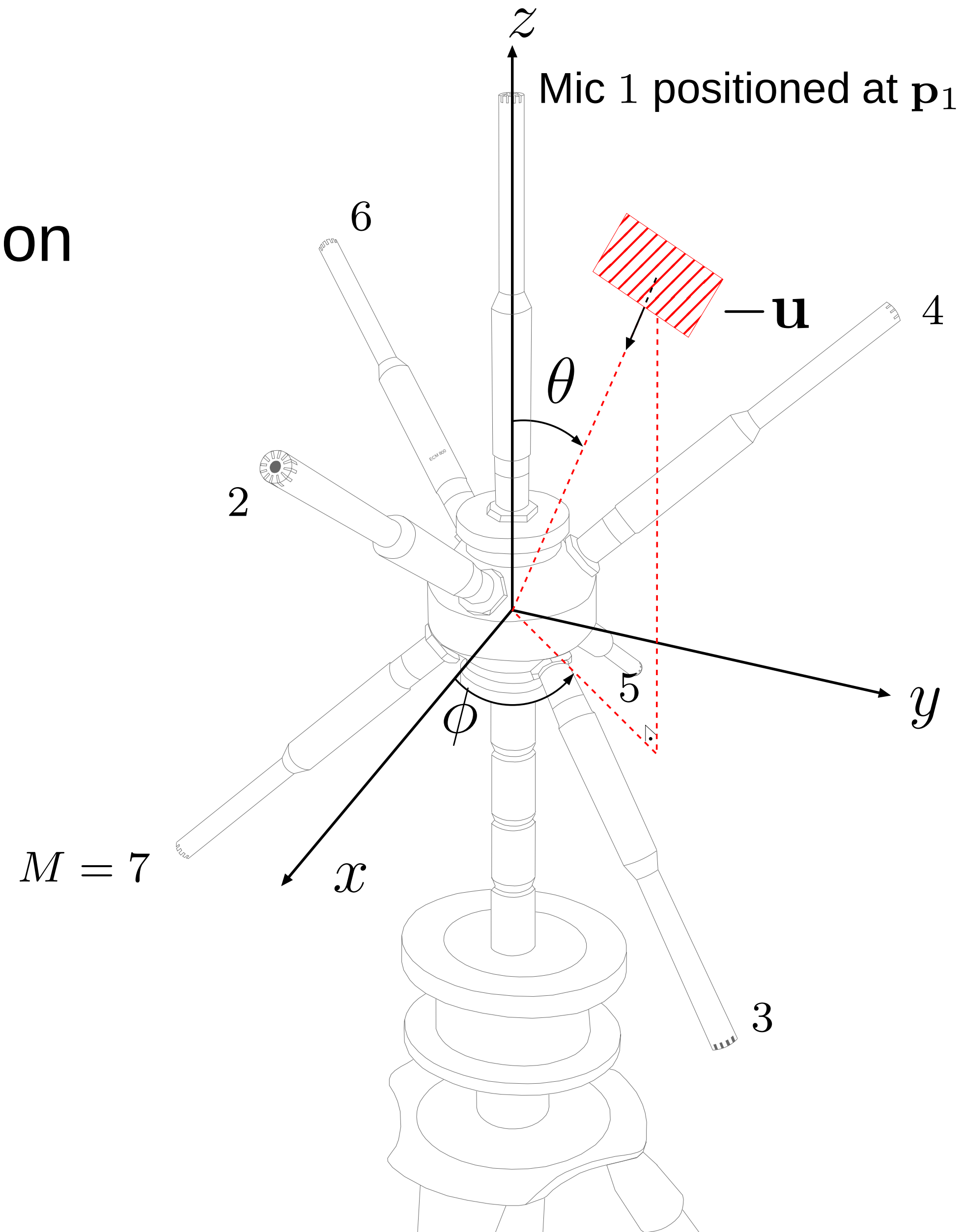

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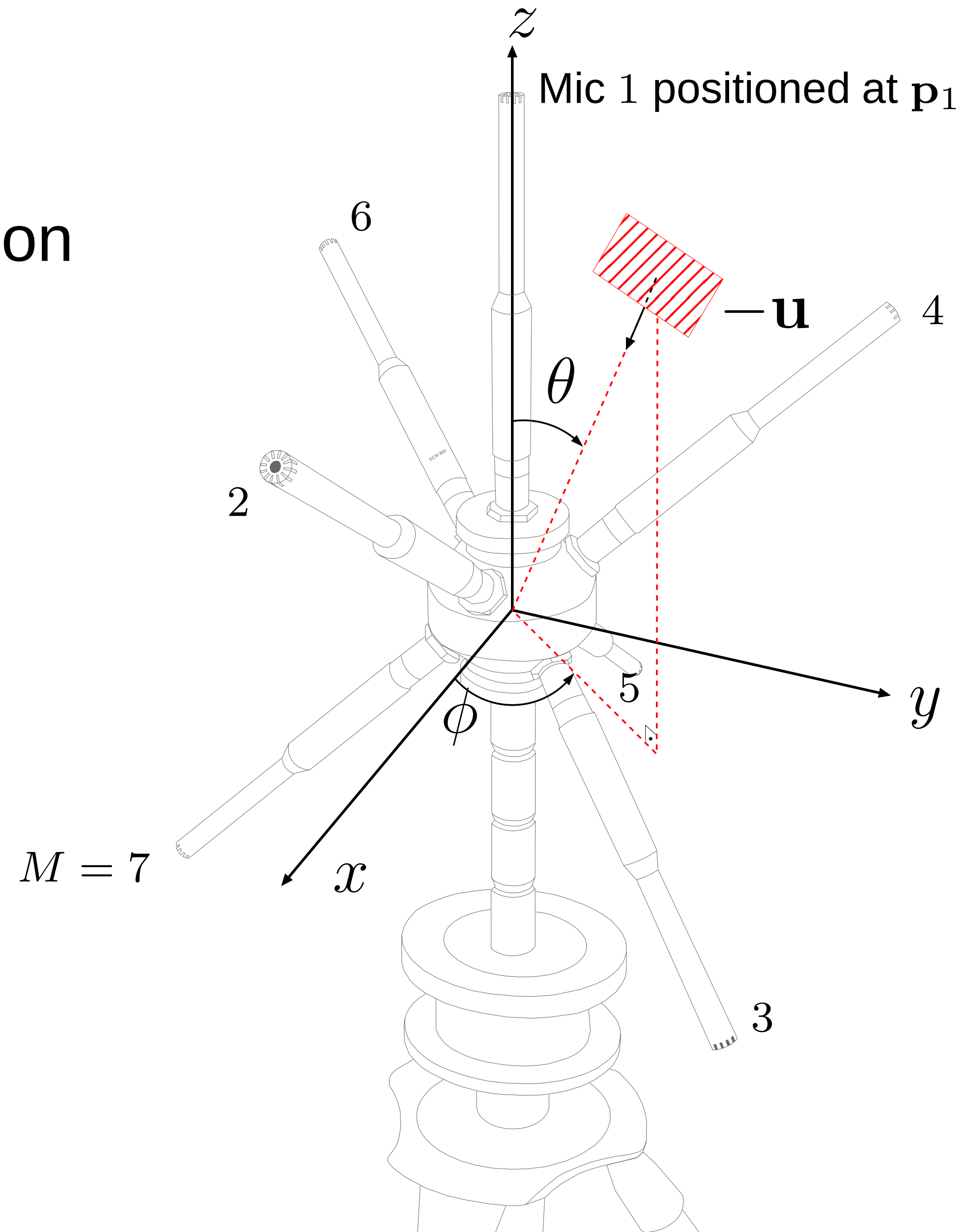

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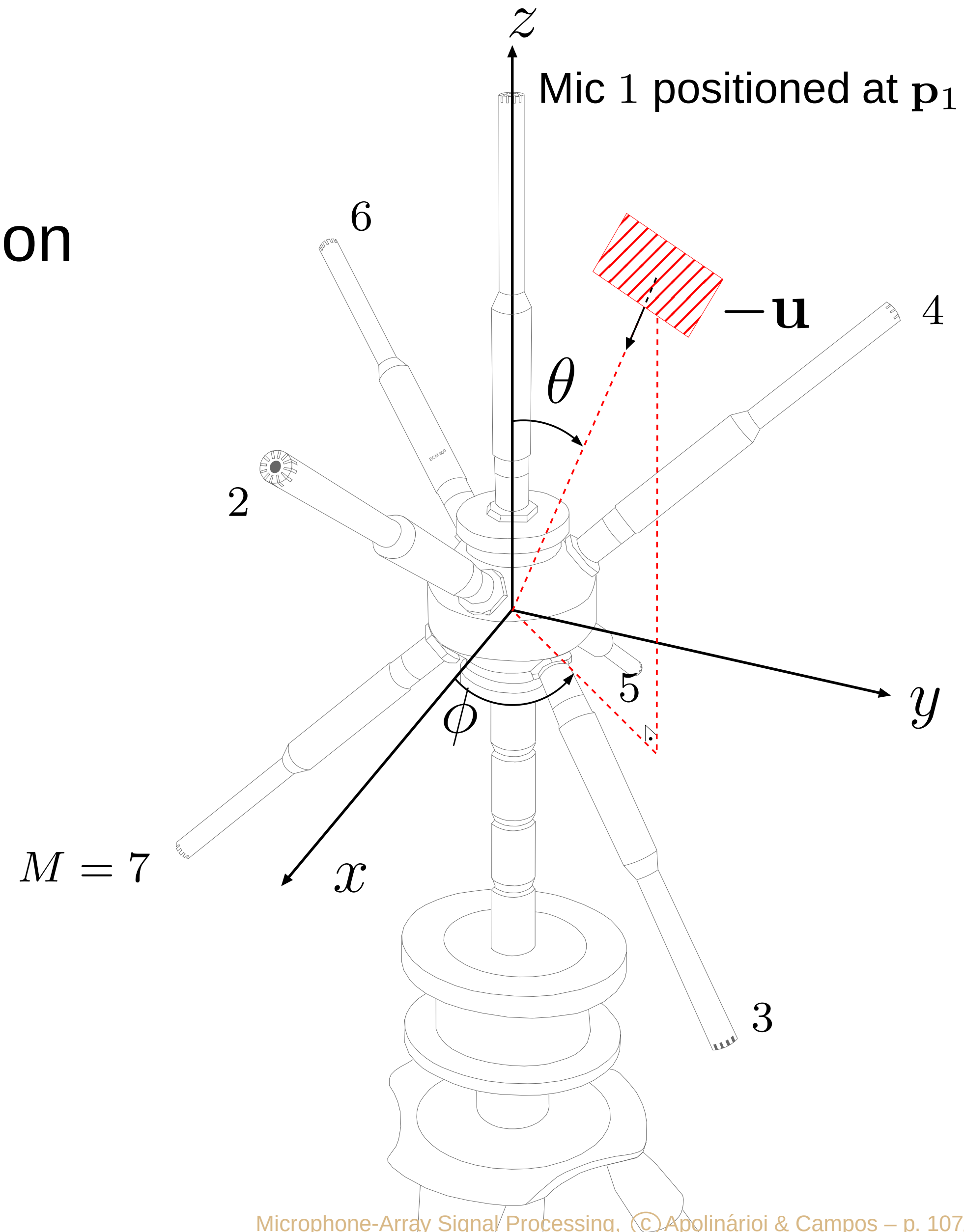
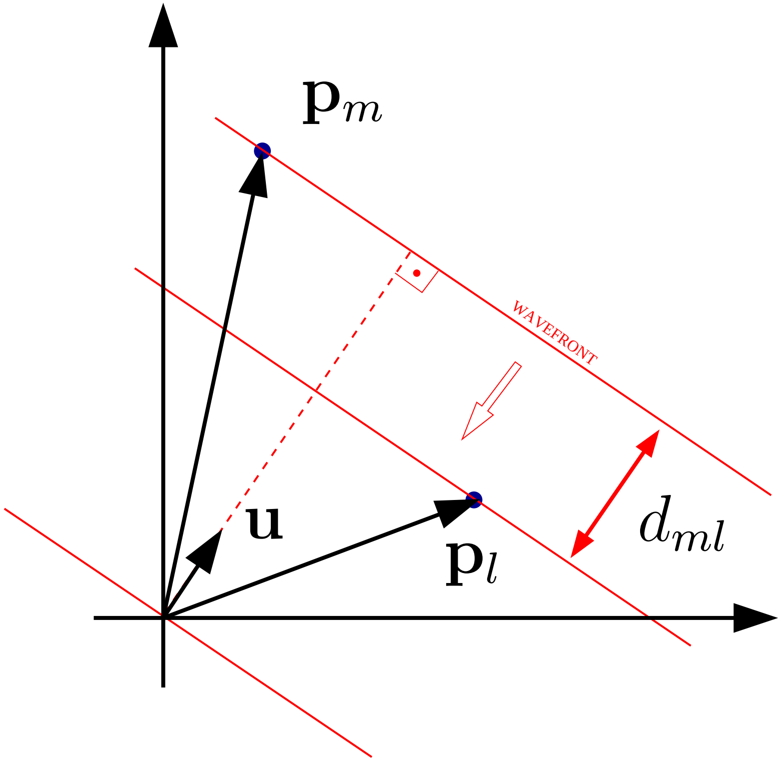


GCC

- We are interested in the TDoA between mics m and l



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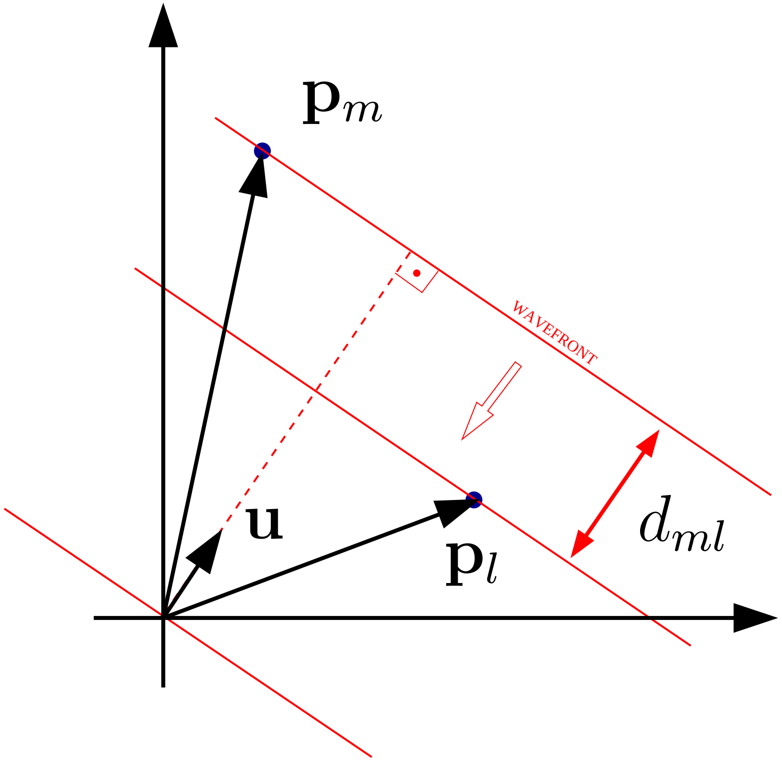

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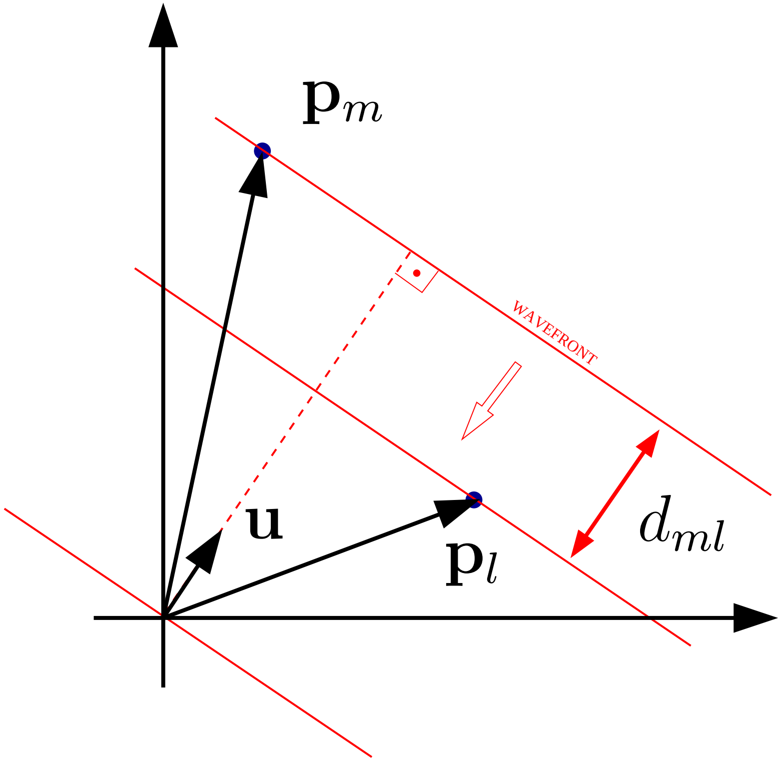

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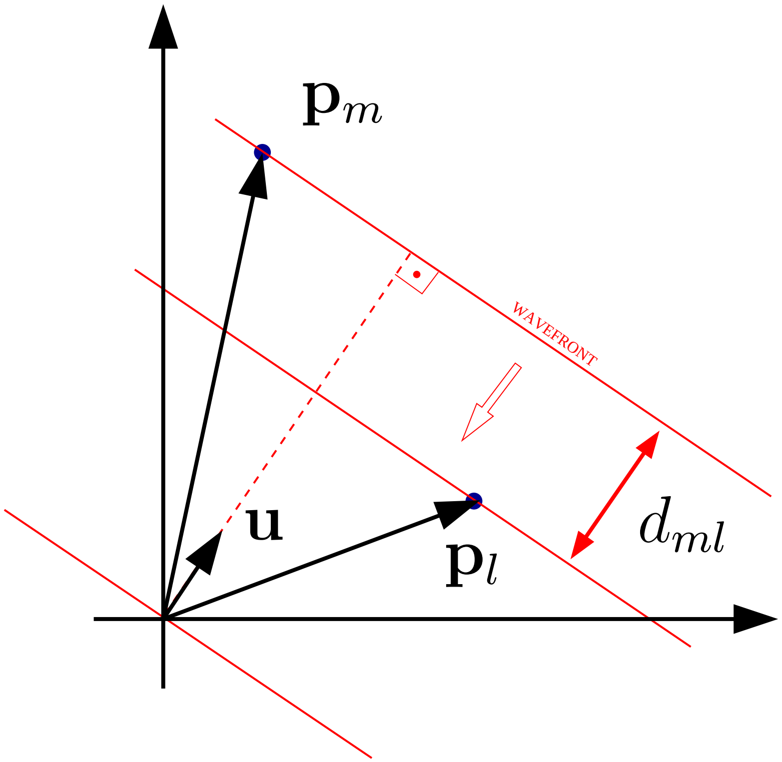

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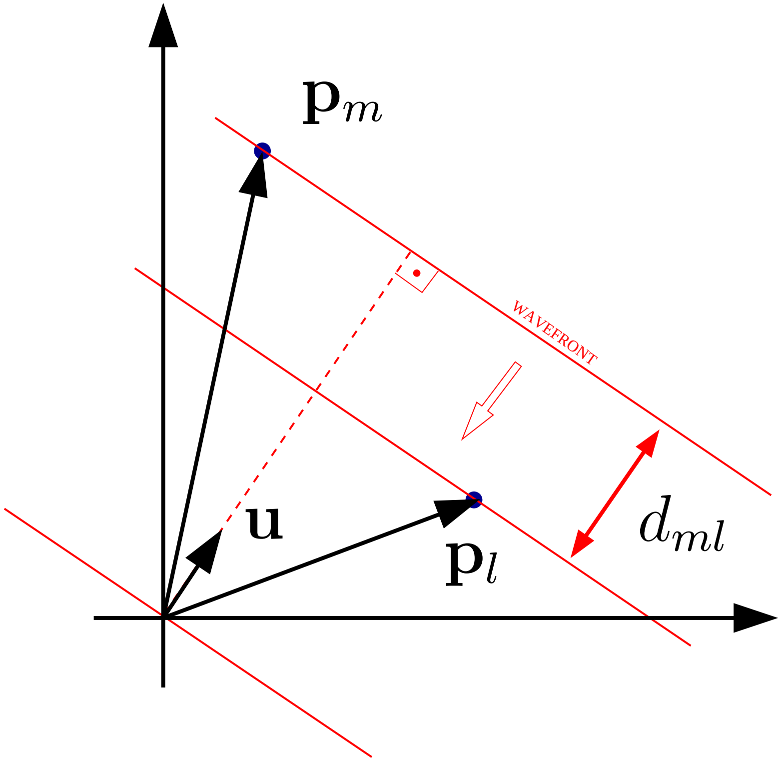
- $r_{x_m x_l}(\tau) = E[x_m(n)x_l(n-\tau)]$



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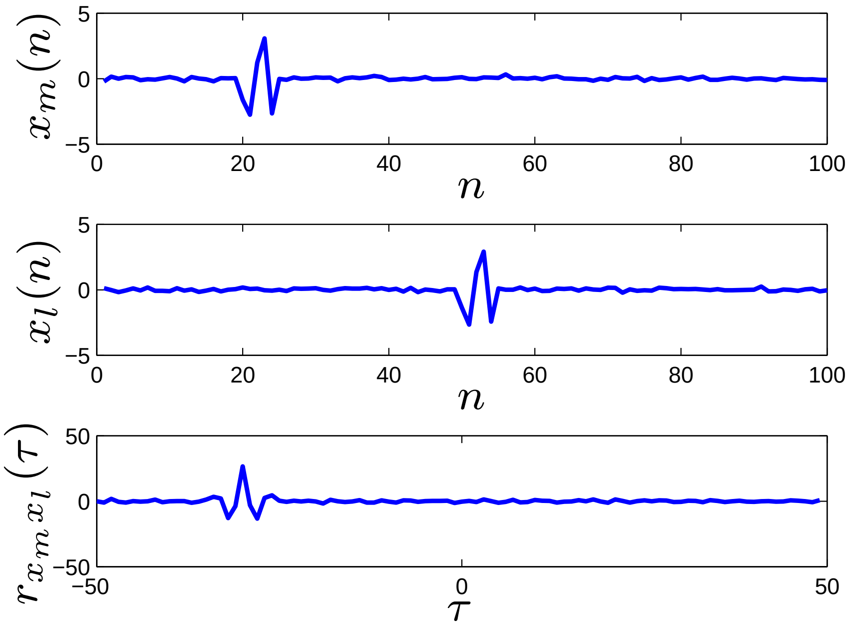

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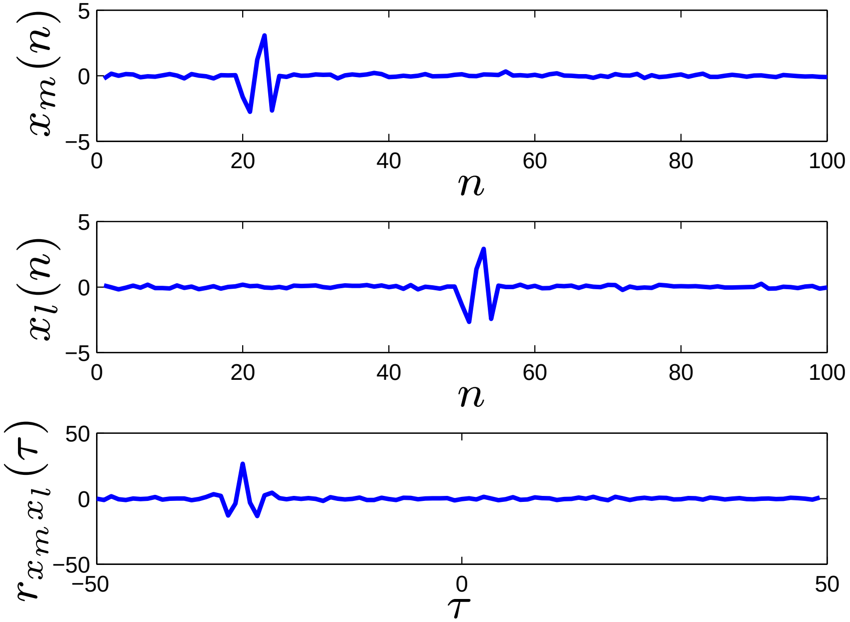
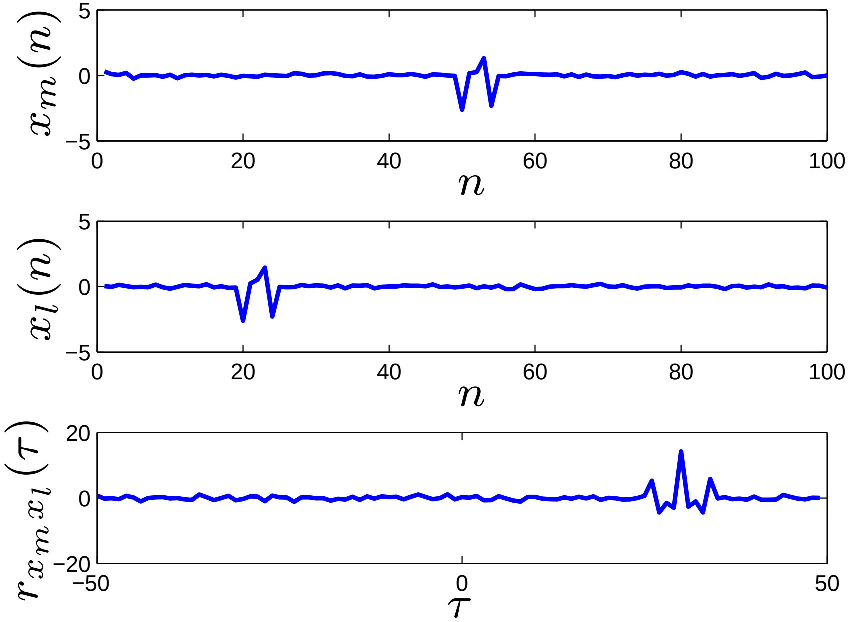


- When it first hits mic l ($\tau_{ml} > 0$):



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$$\hat{r}_{x_m x_l}(\tau) = \sum_{-\infty}^{\infty} x_m(n) x_l(n-\tau) = x_m(\tau) * x_l(-\tau)$$

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- Hence, considering very small additive error and real sequences, we find
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- Which motivates the GCC:

$$\boxed{r_{x_m x_l}^{G}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(\omega) \hat{R}_{x_m x_l}(e^{j\omega}) e^{j\omega\tau} d\omega}$$

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  - $\hat{R}_{x_m}(e^{j\omega}) = |X_m(e^{j\omega})|^2$
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GCC

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GCC

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- Replacing this function in the expression of $r^{G}_{x_mx_l}(\tau)$:
  $r^{PHAT}_{x_mx_l}(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\hat{R}_{x_mx_l}(e^{j\omega})}{|\hat{R}_{x_mx_l}(e^{j\omega})|} e^{j\omega\tau} d\omega$ in which,
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- For the PHAT, in case of having
  $h_m(n) = \alpha_m \delta(n)$ and $h_l(n) = \alpha_l \delta(n - \Delta\tau)$,
  the cross-correlation would be
  $r^{PHAT}_{x_m x_l}(\tau) = \delta(\tau + \Delta\tau) \Rightarrow$ peak in $\tau_{ml} = -\Delta\tau$
  (a perfect indication of a temporal delay!)

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- We then find $\mathbf{u}$ that minimizes ξ by making $\nabla_{\mathbf{u}}\xi = \mathbf{0}$:

  $$\mathbf{A}\mathbf{u} = \mathbf{b}$$

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  and $\mathbf{b} = \bar{\tau}_{12}\Delta\bar{\mathbf{p}}_{12} + \cdots + \bar{\tau}_{(M-1)M}\Delta\bar{\mathbf{p}}_{(M-1)M}$
- And this unit vector is given as $\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \mathbf{A}^{-1}\mathbf{b}$

Azimuth and elevation

- Knowing $\mathbf{u}$ and also the fact that it corresponds to

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- $\cdots$ we compute the azimuth:

$$\boxed{\phi = \arctan\frac{u_y}{u_x}}$$

- And the elevation:

$$\boxed{\text{elevation} = 90^\circ - \theta = 90^\circ - \arccos u_z}$$

Last slide ☺

Thank you!