Microphone-Array Signal Processing

José A. Apolinário Jr. and Marcello L. R. de Campos

{apolin},{mcampos}@ieee.org

COPPE/Poli/UFRJ

IME – Lab. Processamento Digital de Sinais

Outline

1. Introduction and Fundamentals
2. Sensor Arrays and Spatial Filtering
3. Optimal Beamforming
4. Adaptive Beamforming
5. DoA Estimation with Microphone Arrays

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5.0 Signal Preparation

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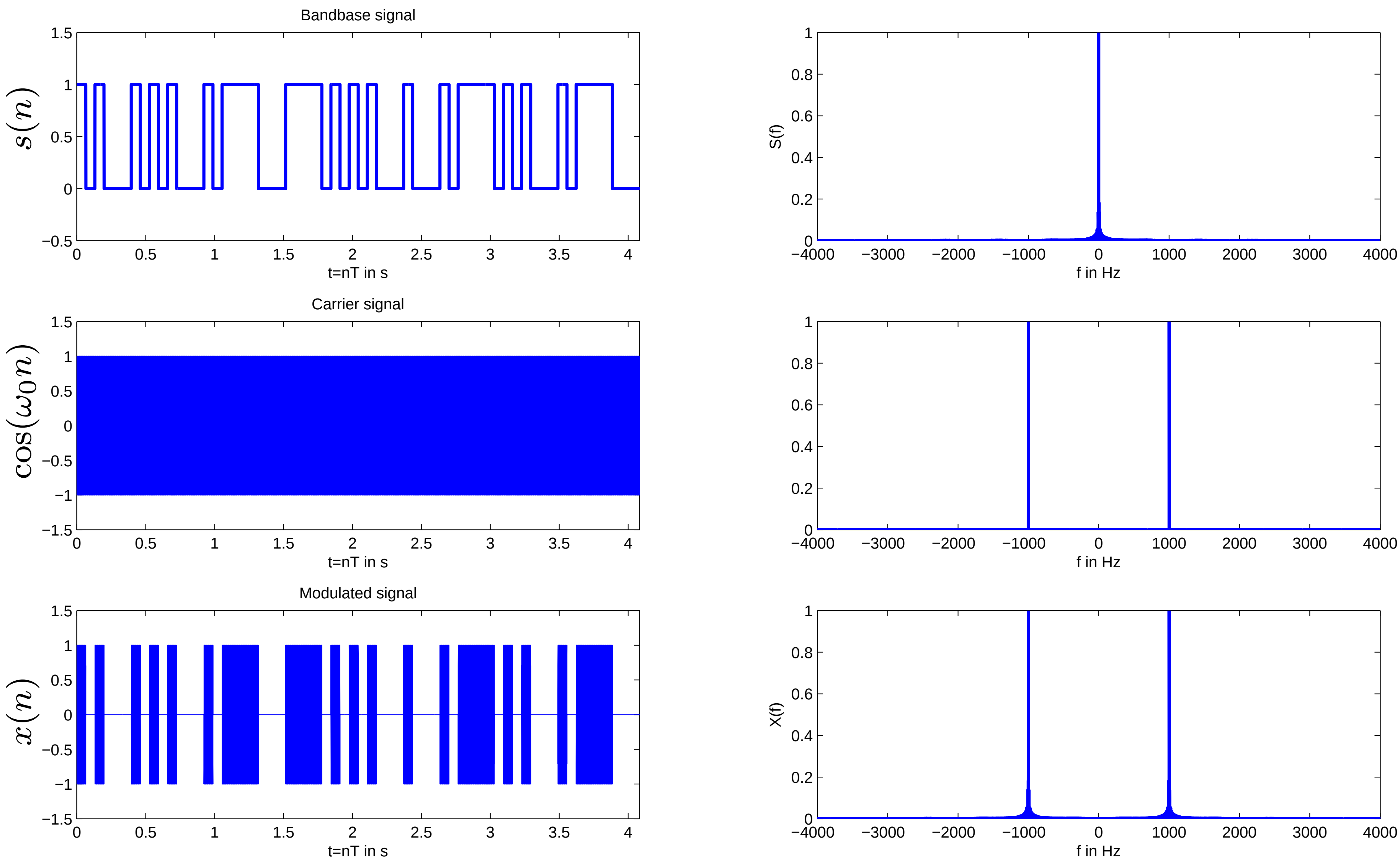
- But, most importantly, the delay is well represented only if the signal is also analytic, i. e., having only non-negative frequency components.
- An analytic signal, mathematically, can be obtained by multiplying its Fourier transform by the continuous Heaviside step function:

$$X_a(e^{j\omega}) = 2X(e^{j\omega})u(\omega), u(\omega) = \begin{cases} 0, \omega < 0 \\ 1, \omega = 0 \\ 1, \omega > 0 \end{cases}$$

- Let $x(n) = s(n)\cos(\omega_0 n)$, $s(n)$ having a maximum frequency component (ω_m) much lower than ω_0:



- If $x(n) = s(n)e^{j\omega_0 n}$, then $x(n)e^{-j\omega_0 \tau} = s(n)e^{j\omega_0 (n-\tau)} \approx x(n-\tau)$ if $\tau \ll 1/\omega_m$

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- We can make
  $$x(n) = s(n)cos(\omega_0 n) = \underbrace{\frac{s(n)}{2}e^{j\omega_0 n}}_{x_+(n)} + \underbrace{\frac{s(n)}{2}e^{-j\omega_0 n}}_{x_-(n)}$$ such that
  $$x(n-\tau) \approx x_+e^{-j\omega_0\tau} + x_-(n)e^{+j\omega_0\tau} = s(n)cos(\omega_0(n-\tau))$$

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- $\cdots$ but, how to obtain $x_+(n)$ or a scaled copy? Using the Hilbert Transform $x_H(n) = \mathcal{HT}\{x(n)\}$ where

$$X_H(e^{j\omega}) = \begin{cases} jX(e^{j\omega}), -\pi < \omega < 0 \\ X(e^{j\omega}), \omega = 0 \\ -jX(e^{j\omega}), 0 < \omega < \pi \end{cases}$$

- Knowing that $x(n) = x_-(n) + x_+(n) = \mathcal{F}^{-1}\left\{X_-(e^{j\omega}) + X_+(e^{j\omega})\right\}$, we compute $y(n) = x(n) + jx_H(n)$

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- Therefore $y(n) = 2\mathcal{F}^{-1}\{X_+(e^{j\omega})\} = s(n)e^{j\omega_0 n}$ which is analytic!

Signal Model

- Consider $x_m(t)$ the signal from the m-th microphone (prior to the A/D converter) corresponding to audio from D sources (directions θ_1 to θ_D) plus noise:
  $x_m(t) = s_1(t - \bar{\tau}_m(\theta_1)) + \cdots + s_D(t - \bar{\tau}_m(\theta_D)) + n_m(t)$

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- For an array with M microphones, we would have:

$$\underbrace{\mathbf{x}(n)}_{M\times 1} = \underbrace{\mathbf{A}}_{M\times D}\underbrace{\mathbf{s}(n)}_{D\times 1} + \underbrace{\mathbf{n}(n)}_{M\times 1}$$

5.1 Signal model

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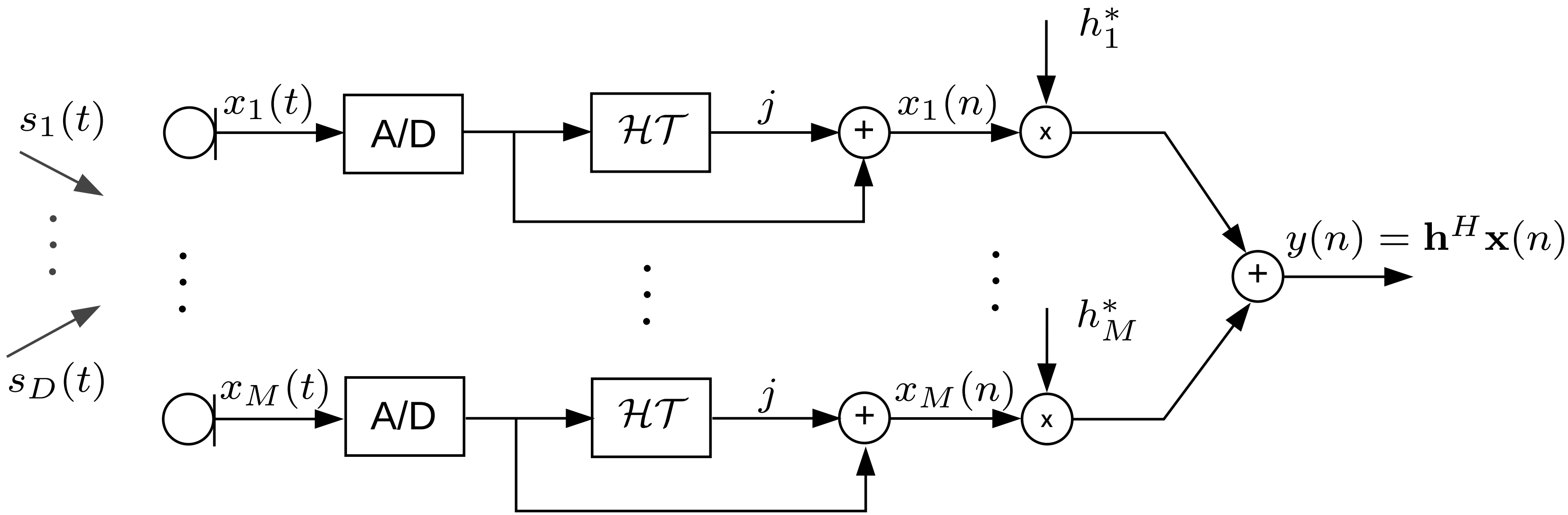

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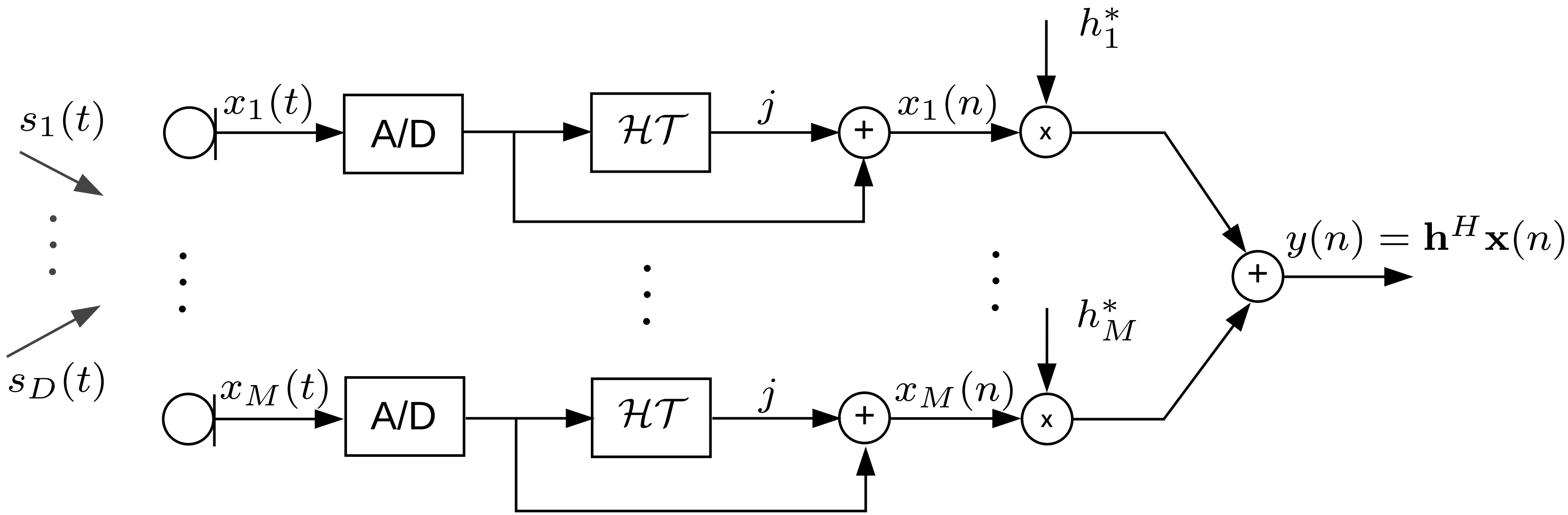

- $$\mathbf{x}(n) = \begin{bmatrix} e^{-j\omega_0 \tau_1(\theta_1)} s_1(n) + \cdots + e^{-j\omega_0 \tau_1(\theta_D)} s_D(n) + n_1(n) \\ \vdots \\ e^{-j\omega_0 \tau_M(\theta_1)} s_1(n) + \cdots + e^{-j\omega_0 \tau_M(\theta_D)} s_D(n) + n_M(n) \end{bmatrix}$$

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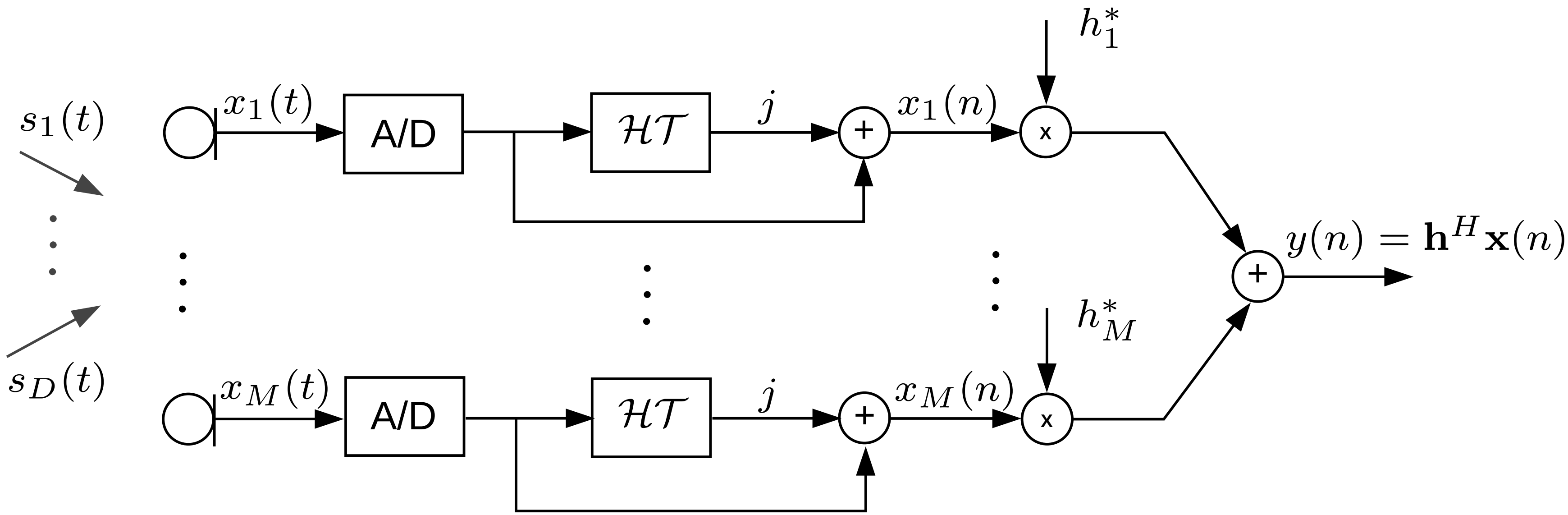

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- Such that the output signal can be written as
$y(n) = \mathbf{h}^H\mathbf{x}(n) = \mathbf{h}^H\left[\mathbf{A}\mathbf{s}(n) + \mathbf{n}(n)\right]$

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- Also note that $E[|y(n)|^2] = \mathbf{h}^H\mathbf{R}_x\mathbf{h}$, $\mathbf{R}_x = E[\mathbf{x}(n)\mathbf{x}^H(n)]$

5.2 Non-parametric methods: BF (beamforming a.k.a. Delay & Sum) and Capon

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  and equating to zero, is $\mathbf{h} = \mathbf{a}(\theta)/M$ which leads to $E[|y(n)|^2] = \frac{\mathbf{a}^H(\theta)\mathbf{R}_x\mathbf{a}(\theta)}{M^2}$

DS DoA

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- Omitting factor $\frac{1}{M^2}$, we estimate the autocorrelation matrix as $\hat{\mathbf{R}}_x = \frac{1}{N}\sum_{n=1}^{N}\mathbf{x}(n)\mathbf{x}^H(n)$ and find the direction of interest by varying θ and obtaining the peak in $\boxed{P_{DS}(\theta) = \mathbf{a}^H(\theta)\hat{\mathbf{R}}_x\mathbf{a}(\theta)}$

Capon

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- Replacing the above coefficient vector in $E[|y(n)|^2]$, we obtain $E[|y(n)|^2] = \frac{1}{\mathbf{a}^H(\theta)\mathbf{R}_x^{-1}\mathbf{a}(\theta)}$

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- Replacing the above coefficient vector in $E[|y(n)|^2]$, we obtain $E[|y(n)|^2] = \frac{1}{\mathbf{a}^H(\theta)\mathbf{R}_x^{-1}\mathbf{a}(\theta)}$
- Therefore, in the Capon DoA, we estimate $\hat{\mathbf{R}}_x = \frac{1}{N}\sum_{n=1}^{N} \mathbf{x}(n)\mathbf{x}^H(n)$ and find the direction of interest by varying θ and obtaining the peak in

$$\boxed{P_{CAPON}(\theta) = \frac{1}{\mathbf{a}^H(\theta)\hat{\mathbf{R}}_x^{-1}\mathbf{a}(\theta)}}$$

5.3 Eigenvalue-Based DoA

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- Also note that $\mathbf{A}$ is $M \times D$, $\mathbf{s}$ is $D \times 1$, and $\mathbf{n}(n)$ is $M \times 1$
- We then write $\mathbf{R}_x = E\left[\mathbf{x}(n)\mathbf{x}^H(n)\right] = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{R}_n$, this last matrix becoming $\mathbf{R}_n = \sigma_n^2\mathbf{I}$ when assuming spatially white noise; $\mathbf{R}_s$ is the $D \times D$ autocorrelation matrix of the signal vector, i.e., $E\left[\mathbf{s}(n)\mathbf{s}^H(n)\right]$

- $\mathbf{R}_x = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{R}_n$ with $D < M$ implies that $\mathbf{A}\mathbf{R}_s\mathbf{A}^H$ is singular (rank D), its determinant is equal to zero and, therefore, $\det\left[\mathbf{R}_x - \sigma_n^2\mathbf{I}\right] = 0$ and σ_n^2 is a (minimum) eigenvalue with multiplicity $M - D$

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- Spectral decomposition of matrix $\mathbf{R}_x$: vector $\mathbf{e}_m$ being an eigenvector of $\mathbf{R}_x$ means that $\mathbf{R}_x\mathbf{e}_m = \lambda_m\mathbf{e}_m$. Collecting all eigenvectors in matrix $\mathbf{E}$, we may write $\mathbf{R}_x\mathbf{E} = \mathbf{E}\boldsymbol{\Lambda} = \left[\mathbf{e}_1 \cdots \mathbf{e}_M\right]$ diag $\left\{\left[\lambda_1 \cdots \lambda_M\right]\right\}$
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- Dividing matrix $\mathbf{E}$ in two parts, the first D columns and the last $N = M - D$ columns, we have:
$$\mathbf{E} = [\underbrace{\mathbf{e}_1 \cdots \mathbf{e}_D}_{\mathbf{E}_S}\, \underbrace{\mathbf{e}_{D+1} \cdots \mathbf{e}_M}_{\mathbf{E}_N}] = [\mathbf{E}_S \;\; \mathbf{E}_N]$$

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- We can find the distance d from a vector $\mathbf{v}$ to the signal subspace $\mathbf{E}_S$ by obtaining $\mathbf{x}$ that minimizes $d = |\mathbf{v} - \mathbf{E}_S\mathbf{x}|$; the result is $d^2 = \mathbf{v}^H\mathbf{E}_N\mathbf{E}_N^H\mathbf{v}$

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- When θ belongs to $\{\theta_1 \cdots \theta_D\}$, this distance should be close to zero
- Its inverse will present peaks. In algorithm MUSIC, we estimate D from the eigenvalues of $\hat{\mathbf{R}}_x$; from its eigenvectors, we form $\mathbf{E}_S$ and $\mathbf{E}_N$, and by varying θ, we shall find peaks in the directions of θ_1 to θ_D in

$$P_{MUSIC}(\theta) = \frac{1}{d^2_{\mathbf{a}(\theta)}} = \frac{1}{\mathbf{a}^H(\theta)\mathbf{E}_N\mathbf{E}_N^H\mathbf{a}(\theta)}$$

MUSIC

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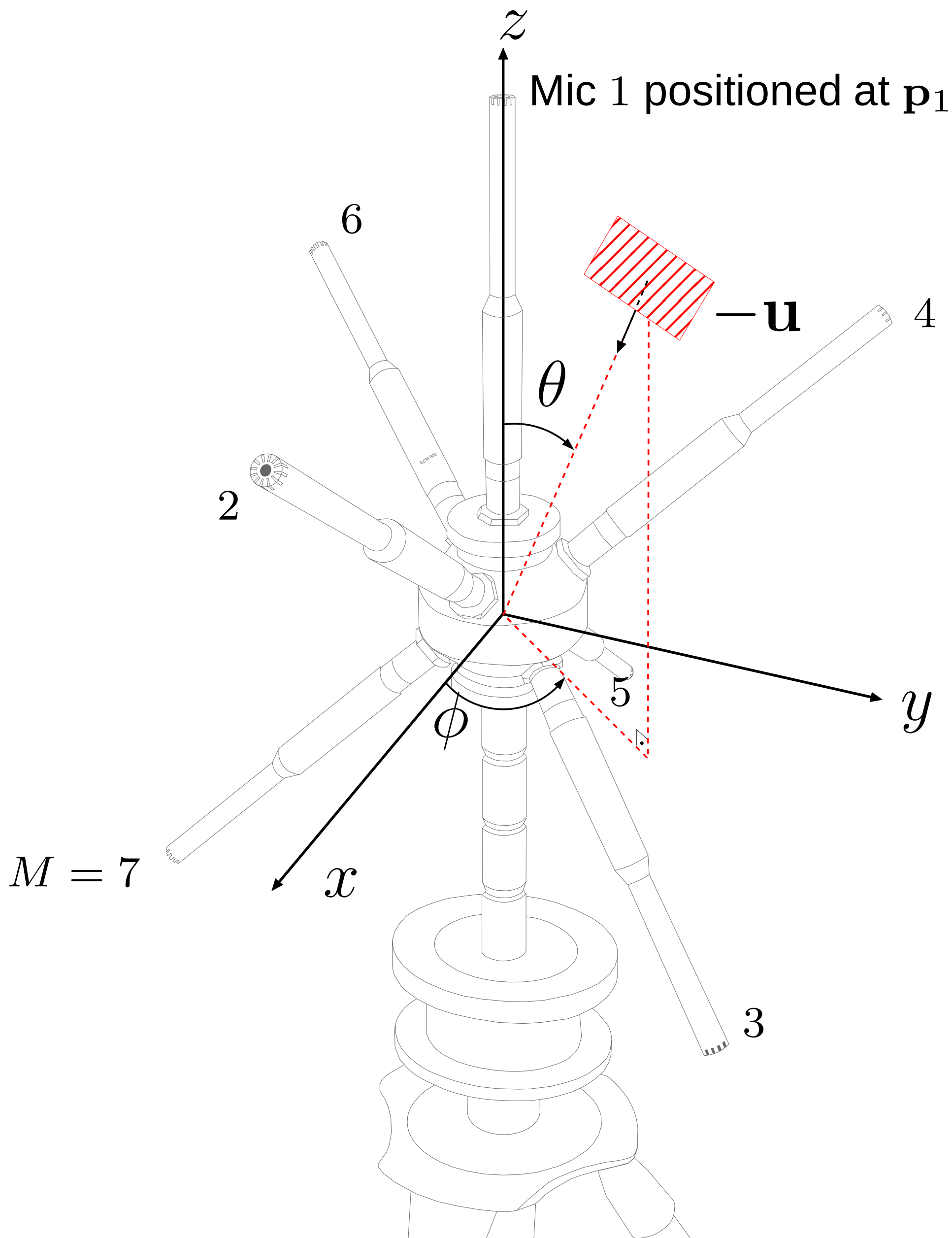
$$\boxed{P_{MUSIC}(\theta) = \frac{1}{d^2_{\mathbf{a}(\theta)}} = \frac{1}{\mathbf{a}^H(\theta)\mathbf{E}_N\mathbf{E}_N^H\mathbf{a}(\theta)}}$$

- If $\mathbf{R}_S$ is required, we compute

$$\mathbf{R}_S = \left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H\left(\mathbf{R}_x - \sigma_n^2\mathbf{I}\right)\mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}$$

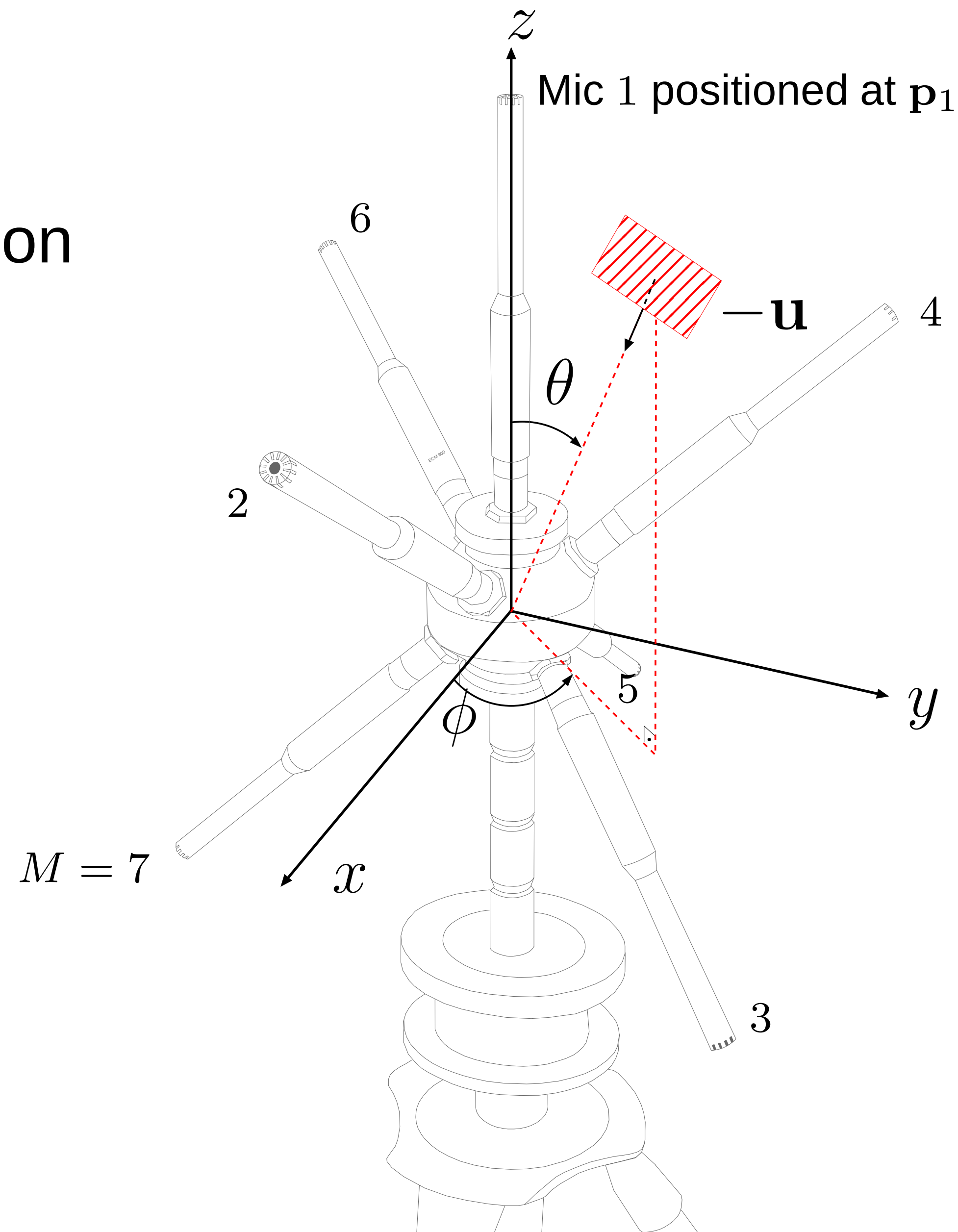
5.4 GCC-Based DoA

- M microphones of an array are in positions $\mathbf{p}_1$ to $\mathbf{p}_M$:



- M microphones of an array are in positions $\mathbf{p}_1$ to $\mathbf{p}_M$:

- $-\mathbf{u}$: unit vector in the direction of propagation



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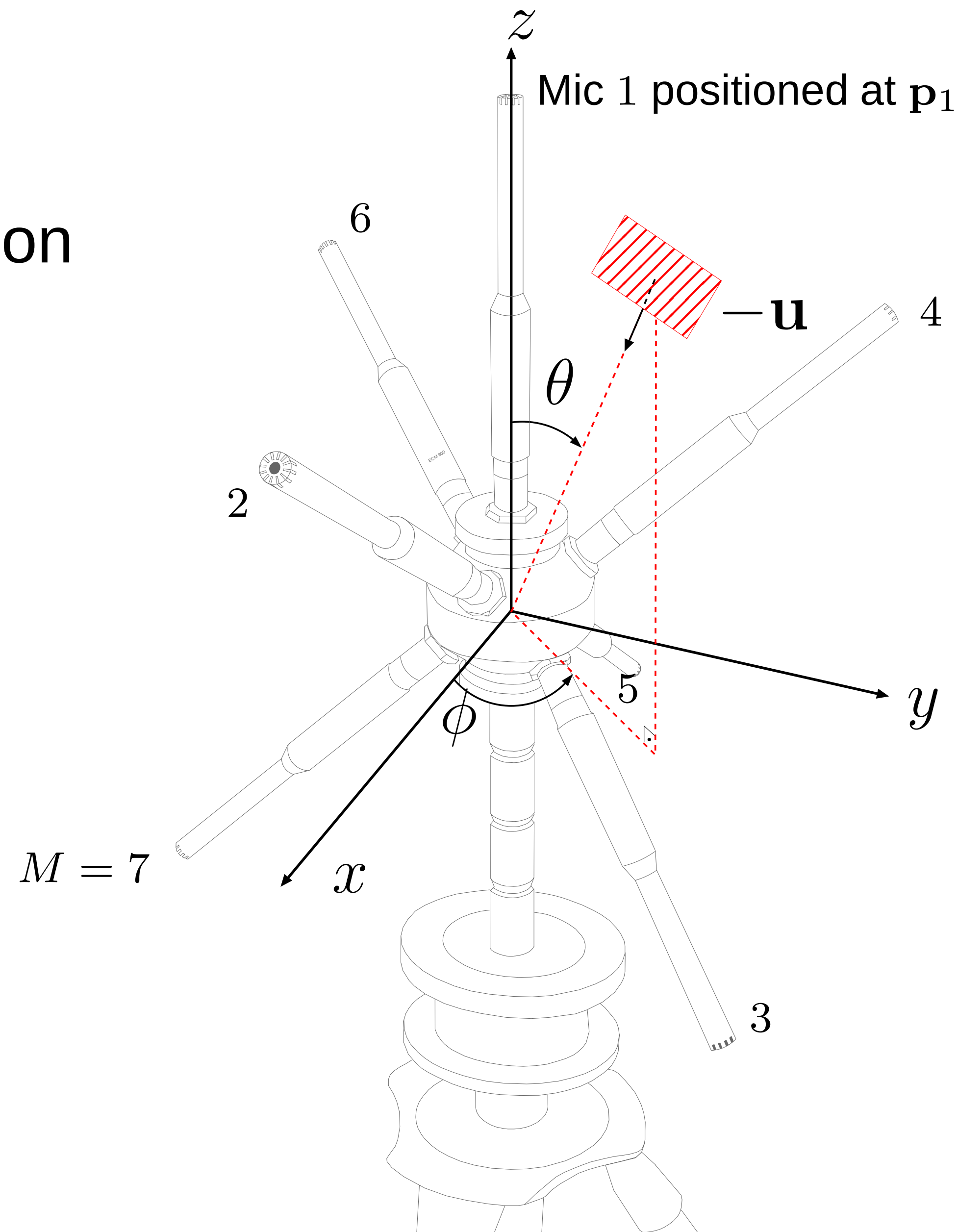

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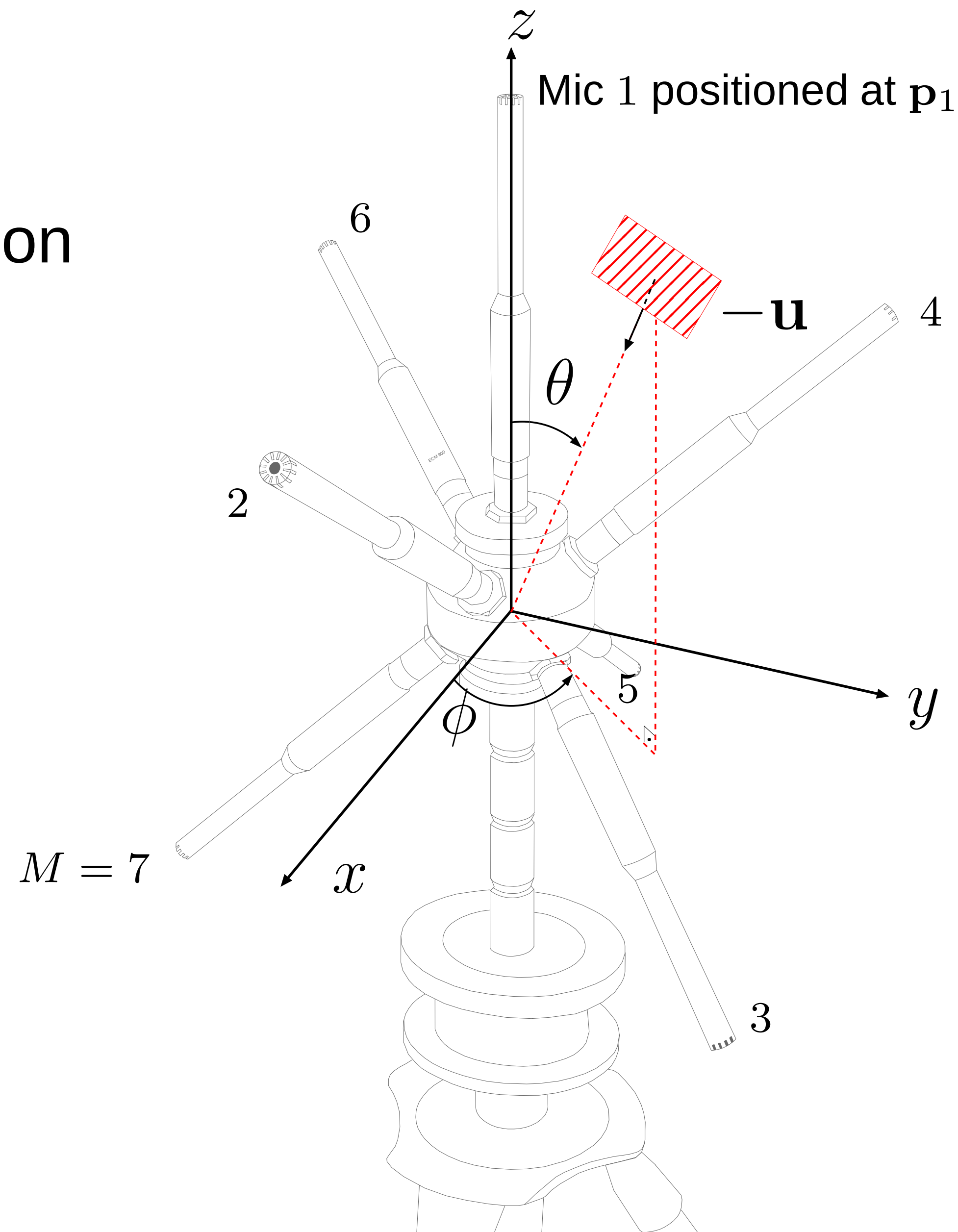

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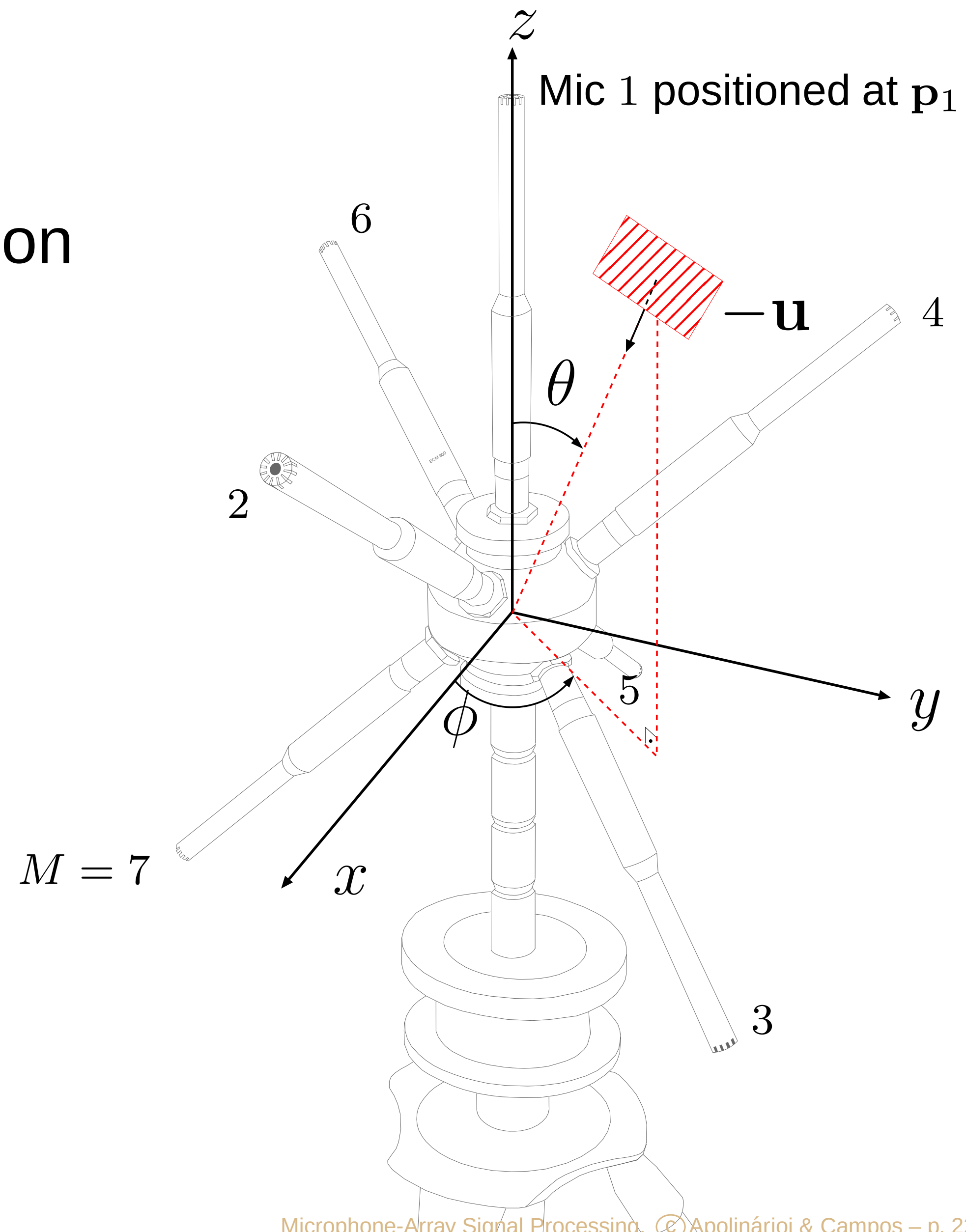

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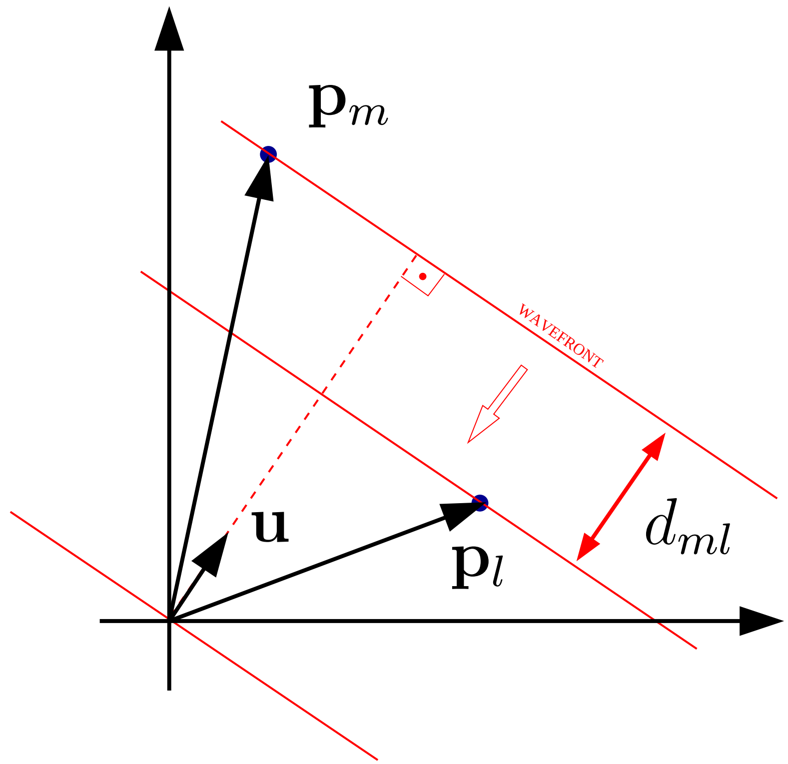

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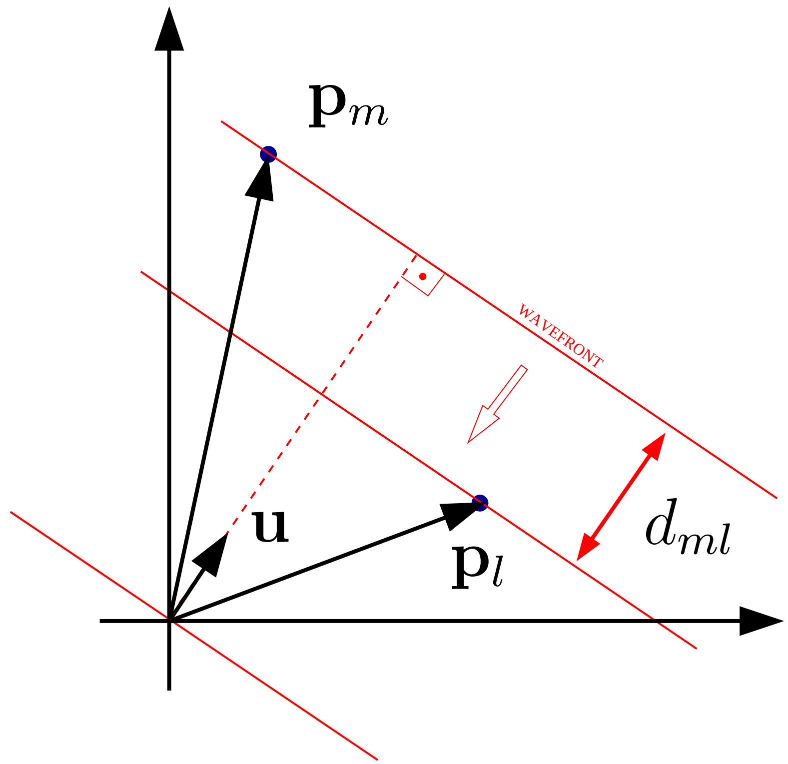

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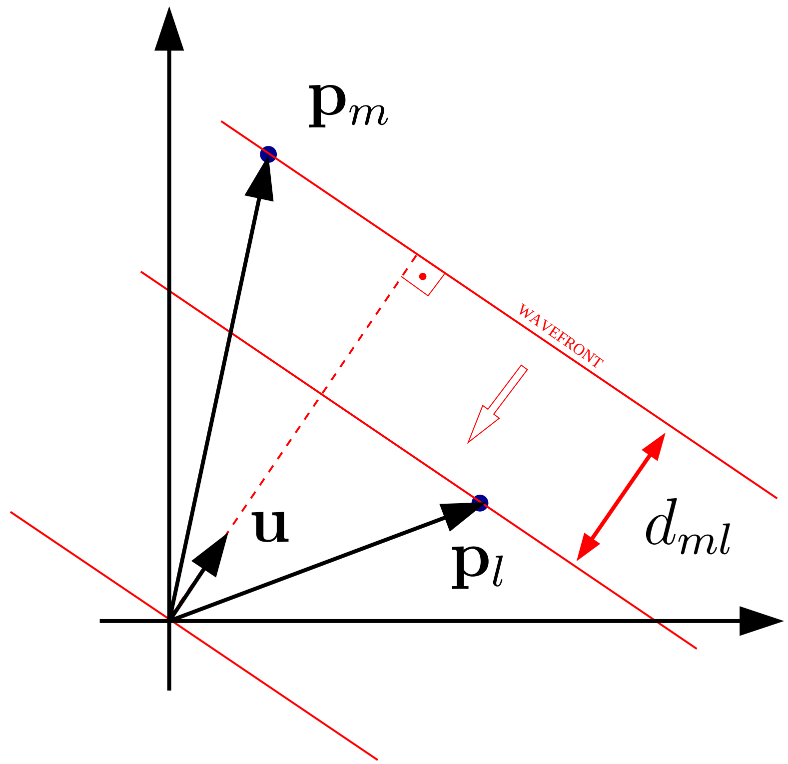

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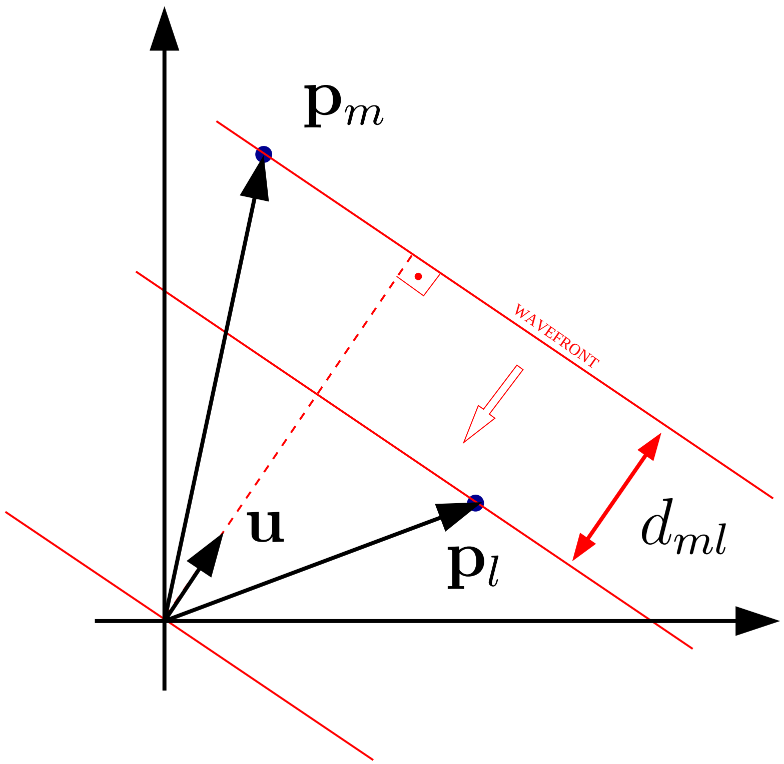

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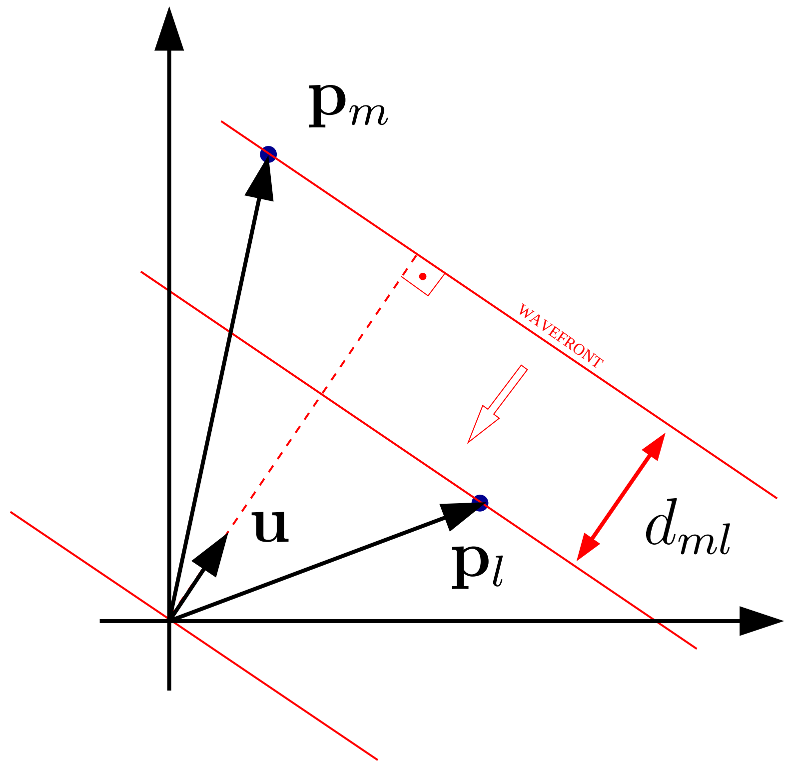
- $r_{x_m x_l}(\tau) = E[x_m(n) x_l(n-\tau)]$



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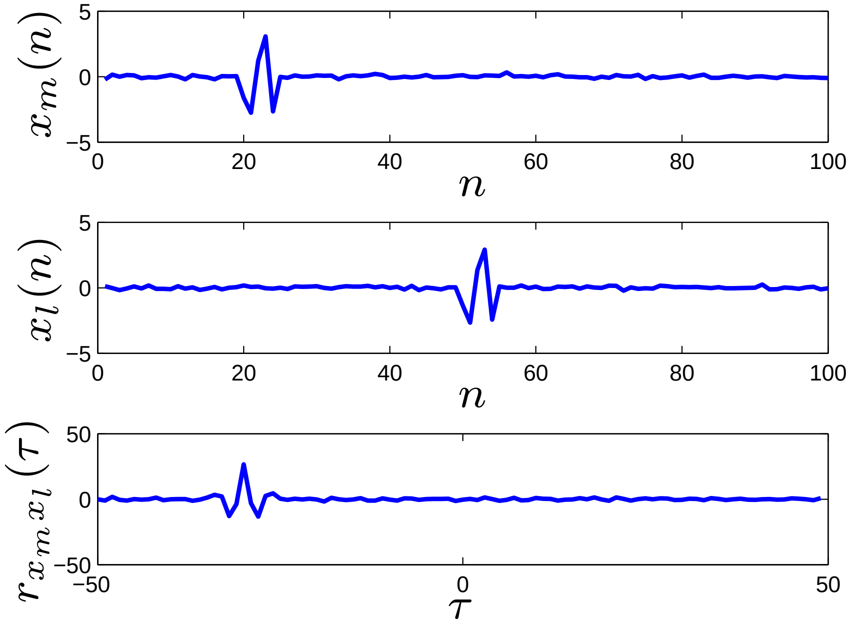

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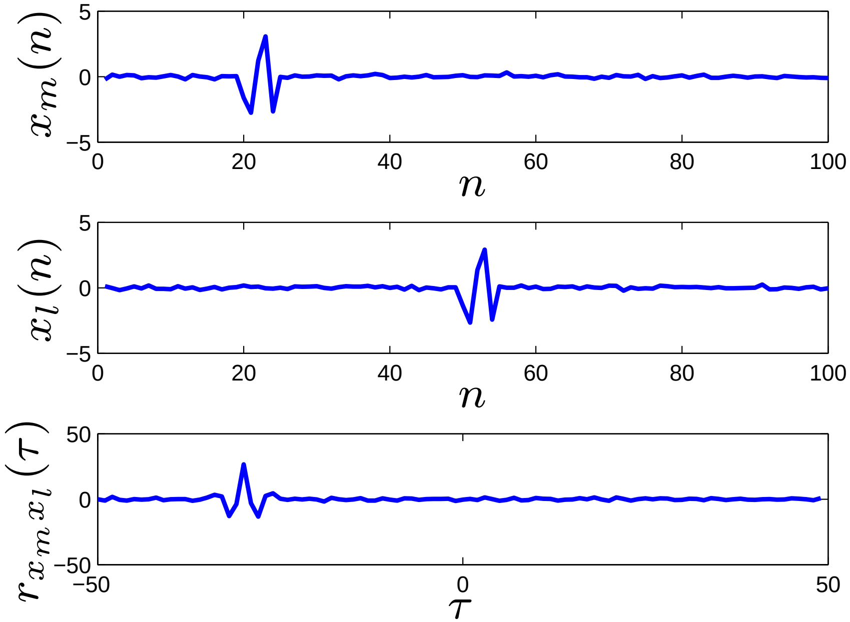

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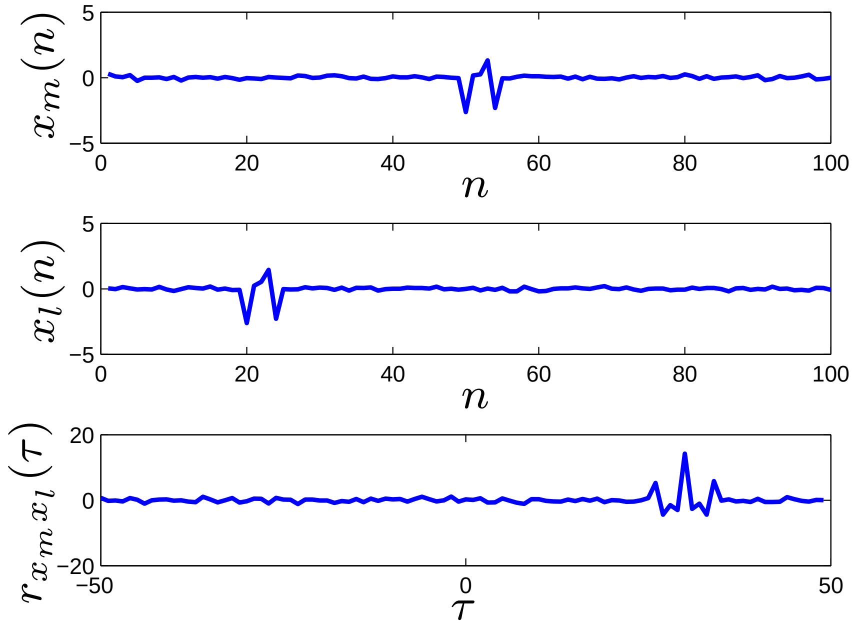

GCC

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- Which motivates the GCC:

$$\boxed{r_{x_m x_l}^{G}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(\omega) \hat{R}_{x_m x_l}(e^{j\omega}) e^{j\omega\tau} d\omega}$$

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  $r^{PHAT}_{x_m x_l}(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\hat{R}_{x_m x_l}(e^{j\omega})}{|\hat{R}_{x_m x_l}(e^{j\omega})|} e^{j\omega\tau} d\omega$ in which,
  after making $\hat{R}_{x_m x_l}(e^{j\omega}) = |S(e^{j\omega})|^2 H_m(e^{j\omega}) H_l^*(e^{j\omega})$,
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- For the PHAT, in case of having
  $h_m(n) = \alpha_m \delta(n)$ and $h_l(n) = \alpha_l \delta(n - \Delta\tau)$,
  the cross-correlation would be
  $r^{PHAT}_{x_m x_l}(\tau) = \delta(\tau + \Delta\tau) \Rightarrow$ peak in $\tau_{ml} = -\Delta\tau$
  (a perfect indication of a temporal delay!)

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- We then find $\mathbf{u}$ that minimizes ξ by making $\nabla_{\mathbf{u}}\xi = \mathbf{0}$:

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where $\mathbf{A} = \Delta\mathbf{p}_{12}\Delta\mathbf{p}_{12}^T + \cdots + \Delta\mathbf{p}_{(M-1)M}\Delta\mathbf{p}_{(M-1)M}^T$
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  and $\mathbf{b} = \bar{\tau}_{12}\Delta\mathbf{p}_{12} + \cdots + \bar{\tau}_{(M-1)M}\Delta\mathbf{p}_{(M-1)M}$
- And this unit vector is given as $\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \mathbf{A}^{-1}\mathbf{b}$

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$$\boxed{\phi = \arctan\frac{u_y}{u_x}}$$

- And the elevation:

$$\boxed{\text{elevation} = 90^\circ - \theta = 90^\circ - \arccos u_z}$$

Thank you!