Microphone-Array Signal Processing

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5.0 Signal Preparation
It is usual to find a delayed signal represented by a multiplication of the signal with exponential $e^{j\omega_0 \tau}$.
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First thing to note: when this is the case, the signal is narrow band with a center frequency in \( \omega_0 \) (in the continuous-time domain, it corresponds to a carrier frequency \( \Omega_0 = f_s \omega_0 \)).

But, most importantly, the delay is well represented only if the signal is also analytic, i.e., having only non-negative frequency components.

An analytic signal, mathematically, can be obtained by multiplying its Fourier transform by the continuous Heaviside step function:

\[
X_a(e^{j\omega}) = 2X(e^{j\omega})u(\omega),
\]

where

\[
u(\omega) = \begin{cases} 
0, & \omega < 0 \\
1, & \omega = 0 \\
1, & \omega > 0 
\end{cases}
\]
Let $x(n) = s(n) \cos(\omega_0 n)$, $s(n)$ having a maximum frequency component ($\omega_m$) much lower than $\omega_0$:
If \( x(n) = s(n)e^{j\omega_0 n} \), then

\[
x(n)e^{-j\omega_0 \tau} = s(n)e^{j\omega_0 (n-\tau)} \approx x(n - \tau) \text{ if } \tau \ll 1/\omega_m
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We can make
\[
x(n) = s(n)\cos(\omega_0 n) = \frac{s(n)}{2}e^{j\omega_0 n} + \frac{s(n)}{2}e^{-j\omega_0 n} \text{ such that } \]
\[
x(n - \tau) \approx x_+(n)e^{-j\omega_0 \tau} + x_-(n)e^{+j\omega_0 \tau} = s(n)\cos(\omega_0(n - \tau))
\]
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\]

\[
\ldots \quad \text{but, how to obtain } x_+(n) \text{ or a scaled copy? Using the Hilbert Transform } \quad x_H(n) = \mathcal{HT}\{x(n)\} \quad \text{where}
\]
\[
X_H(e^{j\omega}) = \begin{cases} 
  jX(e^{j\omega}), & -\pi < \omega < 0 \\
  X(e^{j\omega}), & \omega = 0 \\
  -jX(e^{j\omega}), & 0 < \omega < \pi
\end{cases}
\]
Knowing that
\[ x(n) = x_-(n) + x_+(n) = \mathcal{F}^{-1} \left\{ X_-(e^{j\omega}) + X_+(e^{j\omega}) \right\}, \]
we compute
\[ y(n) = x(n) + jx_H(n). \]
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\[ y(n) = \mathcal{F}^{-1} \{ X_-(e^{j\omega}) + X_+(e^{j\omega}) + j[jX(e^{j\omega}) - jX_+(e^{j\omega})] \} \]

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Therefore
\[ y(n) = 2\mathcal{F}^{-1} \{ X_+(e^{j\omega}) \} = s(n)e^{j\omega_0n} \text{ which is analytic!} \]
Consider $x_m(t)$ the signal from the $m$-th microphone (prior to the A/D converter) corresponding to audio from $D$ sources (directions $\theta_1$ to $\theta_D$) plus noise:

$$x_m(t) = s_1(t - \tilde{\tau}_m(\theta_1)) + \cdots + s_D(t - \tilde{\tau}_m(\theta_D)) + n_m(t)$$
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Assuming \( \bar{\tau}_m(\theta_d) = T\tau_m(\theta_d) \) in \( s \) (\( \tau_m(\theta_d) \) in number of samples), after the A/D converter and \( \{.\} + j\mathcal{H}\mathcal{T}\{.\} \) to make it an analytic signal, we could write

\[
x_m(n) = s_1(n)e^{-j\omega_0\tau_m(\theta_1)} + \cdots + s_D(n)e^{-j\omega_0\tau_m(\theta_D)} + n_m(n)
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$$x_m(n) = s_1(n)e^{-j\omega_0 \tau_m(\theta_1)} + \cdots + s_D(n)e^{-j\omega_0 \tau_m(\theta_D)} + n_m(n)$$

For an array with $M$ microphones, we would have:

$$\mathbf{x}(n) = \mathbf{A} \mathbf{s}(n) + \mathbf{n}(n)$$

$$\begin{bmatrix} M \times 1 \\ M \times D \\ D \times 1 \end{bmatrix} \begin{bmatrix} M \times 1 \\ M \times D \end{bmatrix} \begin{bmatrix} M \times 1 \\ M \times D \end{bmatrix}$$
5.1 Signal model
Assume, initially, we have $D$ narrowband signals coming from unknown directions:
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\[ y(n) = h^H x(n) \]

\[ x(n) = \begin{bmatrix}
  e^{-j\omega_0 \tau_1(\theta_1)} s_1(n) + \cdots + e^{-j\omega_0 \tau_1(\theta_D)} s_D(n) + n_1(n) \\
  \vdots \\
  e^{-j\omega_0 \tau_M(\theta_1)} s_1(n) + \cdots + e^{-j\omega_0 \tau_M(\theta_D)} s_D(n) + n_M(n)
\end{bmatrix} \]
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$$
\begin{align*}
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    \vdots \\
    e^{-j\omega_0 \tau_M(\theta_1)} s_1(n) + \cdots + e^{-j\omega_0 \tau_M(\theta_D)} s_D(n) + n_M(n)
\end{bmatrix}
\end{align*}
$$

Such that the output signal can be written as

$$
y(n) = \mathbf{h}^H \mathbf{x}(n) = \mathbf{h}^H \left[ \mathbf{A} \mathbf{s}(n) + \mathbf{n}(n) \right]
$$
If we now assume one single signal, \( s(n) \), coming from direction \( \theta \), then
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x(n) = s(n) a(\theta) + n(n)
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x(n) = s(n)a(\theta) + n(n)
\]

And the output signal becomes
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y(n) = h^H a(\theta)s(n) + h^H n(n)
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If we make \( h^H a(\theta) = 1 \), the output signal would correspond to
\[
y(n) = s(n) + h^H n(n) \quad \text{(noise)}
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If we make \( h^H a(\theta) = 1 \), the output signal would correspond to \( y(n) = s(n) + h^H n(n) \)

Also note that \( E[|y(n)|^2] = h^H R_x h, \ R_x = E[x(n)x^H(n)] \)
5.2 Non-parametric methods: BF (beamforming a.k.a. Delay & Sum) and Capon
If \( x(n) \) were spatially white, i.e. \( R_x = I \), we would obtain

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Minimizing \( E[|y(n)|^2] = h^H h \) s.t. \( h^H a(\theta) = 1 \), the result, after using Lagrange multiplier, taking the gradient, and equating to zero, is \( h = a(\theta)/M \) which leads to
\[
E[|y(n)|^2] = \frac{a^H(\theta) R_x a(\theta)}{M^2}
\]
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Omitting factor \( \frac{1}{M^2} \), we estimate the autocorrelation matrix as \( \hat{R}_x = \frac{1}{N} \sum_{n=1}^{N} x(n)x^H(n) \) and find the direction of interest by varying \( \theta \) and obtaining the peak in \( P_{DS}(\theta) = a^H(\theta)\hat{R}_xa(\theta) \).
In the method known as Capon, we minimize:

\[ E[|y(n)|^2] = h^H R_x h \text{ subject to } h^H a(\theta) = 1 \]
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Using Lagrange multiplier, we write
\[ \xi = h^H R_x h + \lambda (h^H a(\theta) - 1), \text{ and make } \nabla_h \xi = 0 \text{ such that } h = \frac{R_x^{-1} a(\theta)}{a^H(\theta) R_x^{-1} a(\theta)} \]
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Using Lagrange multiplier, we write

$$\xi = h^H R_x h + \lambda (h^H a(\theta) - 1),$$

and make $\nabla_h \xi = 0$ such that

$$h = \frac{R_x^{-1} a(\theta)}{a^H(\theta) R_x^{-1} a(\theta)}$$

Replacing the above coefficient vector in $E[|y(n)|^2]$, we obtain

$$E[|y(n)|^2] = \frac{1}{a^H(\theta) R_x^{-1} a(\theta)}$$
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\[ E[|y(n)|^2] = h^H R_x h \] subject to \( h^H a(\theta) = 1 \)

Using Lagrange multiplier, we write
\[ \xi = h^H R_x h + \lambda (h^H a(\theta) - 1) \], and make \( \nabla_h \xi = 0 \) such that
\[ h = \frac{R_x^{-1} a(\theta)}{a^H(\theta) R_x^{-1} a(\theta)} \]

Replacing the above coefficient vector in \( E[|y(n)|^2] \), we obtain
\[ E[|y(n)|^2] = \frac{1}{a^H(\theta) R_x^{-1} a(\theta)} \]

Therefore, in the Capon DoA, we estimate
\[ \hat{R}_x = \frac{1}{N} \sum_{n=1}^{N} x(n) x^H(n) \] and find the direction of interest by varying \( \theta \) and obtaining the peak in
\[
P_{CAPON}(\theta) = \frac{1}{a^H(\theta) \hat{R}_x^{-1} a(\theta)}
\]
5.3 Eigenvalue-Based DoA
Coming back to the previous model of $D$ sources, we write $x(n) = As(n) + n(n)$
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We assume $D < M$ (number of signals lower than the number of sensors); this method is known as \textit{parametric} for we make this assumption.
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Also note that $A$ is $M \times D$, $s$ is $D \times 1$, and $n(n)$ is $M \times 1$. 
Coming back to the previous model of $D$ sources, we write $\mathbf{x}(n) = \mathbf{A}\mathbf{s}(n) + \mathbf{n}(n)$

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Also note that $\mathbf{A}$ is $M \times D$, $\mathbf{s}$ is $D \times 1$, and $\mathbf{n}(n)$ is $M \times 1$.

We then write $\mathbf{R}_x = E \left[ \mathbf{x}(n)\mathbf{x}^H(n) \right] = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{R}_n$, this last matrix becoming $\mathbf{R}_n = \sigma_n^2 \mathbf{I}$ when assuming spatially white noise; $\mathbf{R}_s$ is the $D \times D$ autocorrelation matrix of the signal vector, i.e., $E \left[ \mathbf{s}(n)\mathbf{s}^H(n) \right]$.
\( \mathbf{R}_x = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{R}_n \) with \( D < M \) implies that \( \mathbf{A}\mathbf{R}_s\mathbf{A}^H \) is singular (rank \( D \)), its determinant is equal to zero and, therefore, \( \det [\mathbf{R}_x - \sigma_n^2 \mathbf{I}] = 0 \) and \( \sigma_n^2 \) is a (minimum) eigenvalue with multiplicity \( M - D \).
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Spectral decomposition of matrix \( \mathbf{R}_x \): vector \( \mathbf{e}_m \) being an eigenvector of \( \mathbf{R}_x \) means that \( \mathbf{R}_x\mathbf{e}_m = \lambda_m\mathbf{e}_m \).

Collecting all eigenvectors in matrix \( \mathbf{E} \), we may write

\[
\mathbf{R}_x\mathbf{E} = \mathbf{E}\Lambda = [\mathbf{e}_1 \cdots \mathbf{e}_M] \text{diag}\{[\lambda_1 \cdots \lambda_M]\}
\]

\( \Rightarrow \mathbf{R}_x = \mathbf{E}\Lambda\mathbf{E}^H \)
\[ R_x = AR_sA^H + R_n \] with \( D < M \) implies that \( AR_sA^H \) is singular (rank \( D \)), its determinant is equal to zero and, therefore, \( \det [R_x - \sigma_n^2 I] = 0 \) and \( \sigma_n^2 \) is a (minimum) eigenvalue with multiplicity \( M - D \).

Spectral decomposition of matrix \( R_x \): vector \( e_m \) being an eigenvector of \( R_x \) means that \( R_x e_m = \lambda_m e_m \).

Collecting all eigenvectors in matrix \( E \), we may write
\[
R_x E = E \Lambda = [e_1 \cdots e_M] \text{diag} \{[\lambda_1 \cdots \lambda_M]\}
\]
\[ \Rightarrow R_x = E \Lambda E^H \]

Dividing matrix \( E \) in two parts, the first \( D \) columns and the last \( N = M - D \) columns, we have:
\[
E = [\underbrace{e_1 \cdots e_D}_{E_S} \underbrace{e_{D+1} \cdots e_M}_{E_N}] = [E_S \ E_N]
\]
Noting that $EE^H = I$, we can write $E_S E_S^H + E_N E_N^H = I$
Noting that $\mathbf{EE}^H = \mathbf{I}$, we can write $\mathbf{E}_S\mathbf{E}_S^H + \mathbf{E}_N\mathbf{E}_N^H = \mathbf{I}$.

The columns of $\mathbf{E}_S$ span the $D$-dimensional signal subspace while the columns of $\mathbf{E}_N$ span the $N$-dimensional noise subspace.
Noting that $\mathbf{E} \mathbf{E}^H = \mathbf{I}$, we can write $\mathbf{E}_S \mathbf{E}_S^H + \mathbf{E}_N \mathbf{E}_N^H = \mathbf{I}$.

The columns of $\mathbf{E}_S$ span the $D$-dimensional signal subspace while the columns of $\mathbf{E}_N$ span the $N$-dimensional noise subspace.

A vector in the signal subspace is a linear combination of the columns of $\mathbf{E}_S$. An example:

$$\sum_{d=1}^{D} x_d \mathbf{e}_d = \mathbf{E}_S \mathbf{x}, \quad \mathbf{x} = [x_1 \cdots x_D]^T$$
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A vector in the signal subspace is a linear combination of the columns of $\mathbf{E}_S$. An example:

$$\sum_{d=1}^{D} x_d e_d = \mathbf{E}_S \mathbf{x}, \mathbf{x} = [x_1 \cdots x_D]^T$$

We can find the distance $d$ from a vector $\mathbf{v}$ to the signal subspace $\mathbf{E}_S$ by obtaining $\mathbf{x}$ that minimizes $d = |\mathbf{v} - \mathbf{E}_S \mathbf{x}|$; the result is $d^2 = \mathbf{v}^H \mathbf{E}_N \mathbf{E}_N^H \mathbf{v}$. 

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The squared distance from vector \( a(\theta) \) to the signal subspace (spanned by \( E_S \)) is
\[
d^2 = a^H(\theta)E_N E_N^H a(\theta)
\]

When \( \theta \) belongs to \( \{\theta_1 \cdots \theta_D\} \), this distance should be close to zero.

Its inverse will present peaks. In algorithm MUSIC, we estimate \( D \) from the eigenvalues of \( \hat{R}_x \); from its eigenvectors, we form \( E_S \) and \( E_N \), and by varying \( \theta \), we shall find peaks in the directions of \( \theta_1 \) to \( \theta_D \) in

\[
P_{MUSIC}(\theta) = \frac{1}{d^2 a(\theta)} = \frac{1}{a^H(\theta)E_N E_N^H a(\theta)}
\]
The squared distance from vector $a(\theta)$ to the signal subspace (spanned by $E_S$) is $d^2 = a^H(\theta)E_NE_N^H a(\theta)$

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Its inverse will present peaks. In algorithm MUSIC, we estimate $D$ from the eigenvalues of $\hat{R}_x$; from its eigenvectors, we form $E_S$ and $E_N$, and by varying $\theta$, we shall find peaks in the directions of $\theta_1$ to $\theta_D$ in

$$P_{MUSIC}(\theta) = \frac{1}{d^2_{a(\theta)}} = \frac{1}{a^H(\theta)E_NE_N^H a(\theta)}$$

If $R_S$ is required, we compute

$$R_S = (A^H A)^{-1} A^H (R_x - \sigma_n^2 I) A (A^H A)^{-1}$$
5.4 GCC-Based DoA
$M$ microphones of an array are in positions $p_1$ to $p_M$:
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- $\mathbf{u}$: unit vector in the direction of propagation
$M$ microphones of an array are in positions $p_1$ to $p_M$:

- $-\mathbf{u}$: unit vector in the direction of propagation

$\theta$: grazing angle ($\frac{\pi}{2}$ - elevation angle)
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- $u$: unit vector in the direction of propagation

$\theta$: grazing angle ($\frac{\pi}{2}$ - elevation angle)

$\phi$: horizontal angle (azimuth)
$M$ microphones of an array are in positions $p_1$ to $p_M$:

- $u$: unit vector in the direction of propagation

$\theta$: grazing angle
($\frac{\pi}{2}$ - elevation angle)

$\phi$: horizontal angle
(azimuth)

$$u = \begin{bmatrix}
\sin \theta \cos \phi \\
\sin \theta \sin \phi \\
\cos \theta
\end{bmatrix}$$
We are interested in the TDoA between mics $m$ and $l$. 
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TDoA:

$$\tilde{\tau}_{ml} = \frac{d_{ml}}{v_{som}} = \tau_{ml} T = \frac{\tau_{ml}}{f_s}$$
We are interested in the TDoA between mics $m$ and $l$.

Note that $d_{ml} = u^T (p_m - p_l) = \Delta p_{ml}$.

TDoA:

$\bar{\tau}_{ml} = \frac{d_{ml}}{v_{som}} = \tau_{ml} T = \frac{\tau_{ml}}{f_s}$

$\tau_{ml}$ (in number of samples) is to be obtained from the peak of $\hat{r}_{xm,x_l}(\tau)$.
We are interested in the TDoA between mics $m$ and $l$

Note that $d_{ml} = u^T(p_m - p_l)$

TDoA:

$$\bar{\tau}_{ml} = \frac{d_{ml}}{v_{som}} = \tau_{ml} \frac{T}{f_s}$$

$\tau_{ml}$ (in number of samples) is to be obtained from the peak of $\hat{r}_{xm xl}(\tau)$

$$r_{xm xl}(\tau) = E[x_m(n) x_l(n - \tau)]$$
When the sound frontwave first hits microphone \( m \) \( (\tau_{ml} < 0) \):
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When it first hits \( l \) \((\tau_{ml} > 0)\):
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\[ \hat{r}_{x_mx_l}(\tau) = \sum_{-\infty}^{\infty} x_m(n)x_l(n - \tau) = x_m(\tau) * x_l(-\tau) \]
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The cross-power spectrum density (CPSD):
$$\hat{R}_{xm, x_l}(e^{j\omega}) = \mathcal{F}\{x_m(\tau) \ast x_l(-\tau)\} = X_m(e^{j\omega})X_l(e^{-j\omega})$$
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x_m(n) = s(n) * h_m(n) + n_m(n) \text{ and similarly for } x_l(n)
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Hence, considering very small additive error and real sequences, we find
$$\hat{R}_{x_m x_l}(e^{j\omega}) \approx |S(e^{j\omega})|^2 H_m(e^{j\omega})H^*_l(e^{j\omega}) \text{ and}$$
$$\hat{r}_{x_m x_l}(\tau) \approx \frac{1}{2\pi} \int_{-\pi}^{\pi} H_m(e^{j\omega})H^*_l(e^{j\omega})\hat{R}_s(e^{j\omega})e^{j\omega\tau} d\omega$$
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Which motivates the GCC:
\[ r_{x_m x_l}^G(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(\omega) \hat{R}_{x_m x_l}(e^{j\omega})e^{j\omega \tau} d\omega \]
Types of $\psi(\omega)$

- Classical cross-correlation:
  
  \[ \psi(\omega) = 1 \]
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- Maximum Likelihood (ML):
  \[ \psi(\omega) = \frac{|X_m(e^{j\omega})||X_l(e^{j\omega})|}{\hat{R}_{nn}(e^{j\omega})\hat{R}_{xm}(e^{j\omega})+\hat{R}_{nl}(e^{j\omega})\hat{R}_{xl}(e^{j\omega})} \]
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  - $\hat{R}_{xm}(e^{j\omega}) = |X_m(e^{j\omega})|^2$
  
  - $\hat{R}_{xl}(e^{j\omega}) = |X_l(e^{j\omega})|^2$
  
  - $\hat{R}_{nm}(e^{j\omega}) = |N_m(e^{j\omega})|^2$ (estimated during silence interval)
  
  - $\hat{R}_{nl}(e^{j\omega}) = |N_l(e^{j\omega})|^2$ (estimated during silence interval)
PHAT (Phase Transform):

\[ \psi(\omega) = \frac{1}{|\hat{R}_{mm} x_l(e^{j\omega})|} \]
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Replacing this function in the expression of \( r^G_{xmxl}(\tau) \):

\[ r^{PHAT}_{xmxl}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\hat{R}_{xmxl}(e^{j\omega})}{|\hat{R}_{xmxl}(e^{j\omega})|} e^{j\omega \tau} d\omega \]

in which,

after making \( \hat{R}_{xmxl}(e^{j\omega}) = |S(e^{j\omega})|^2 H_m(e^{j\omega}) H_l^*(e^{j\omega}) \),

we have \( r^{PHAT}_{xmxl}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\angle H_m - \angle H_l + \omega \pi)} d\omega \)
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For the PHAT, in case of having
\[
h_m(n) = \alpha_m \delta(n) \text{ and } h_l(n) = \alpha_l \delta(n - \Delta \tau),
\]
the cross-correlation would be
\[
r^{PHAT}_{xm x_l}(\tau) = \delta(\tau + \Delta \tau) \Rightarrow \text{peak in } \tau_{ml} = -\Delta \tau
\]
(a perfect indication of a temporal delay!)
Assuming we have all possible \( M(M - 1)/2 \) delays \( \tau_{ml} \), we want angles \( \phi \) and \( \theta \).
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We define a cost function:

\[
\xi = (\bar{\tau}_{12} - \Delta p_{12}^T u)^2 + \cdots + \left(\bar{\tau}_{(M-1)M} - \Delta p_{(M-1)M}^T u\right)^2
\]

with \(\bar{\tau}_{ml} = \tau_{ml} / f_s\).
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with $\bar{\tau}_{ml} = \tau_{ml}/f_s$

We then find $u$ that minimizes $\xi$ by making $\nabla_u \xi = 0$:

$$
Au = b
$$

where $A = \Delta p_{12}\Delta p_{12}^T + \cdots + \Delta p_{(M-1)M}\Delta p_{(M-1)M}^T$

and $b = \bar{\tau}_{12}\Delta p_{12} + \cdots + \bar{\tau}_{(M-1)M}\Delta p_{(M-1)M}$
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and \( b = \bar{\tau}_{12} \Delta p_{12} + \cdots + \bar{\tau}_{(M-1)M} \Delta p_{(M-1)M} \)

And this unit vector is given as \( u = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = A^{-1} b \)
Knowing \( \mathbf{u} \) and also the fact that it corresponds to
\[
\begin{bmatrix}
\sin \theta \cos \phi \\
\sin \theta \sin \phi \\
\cos \theta
\end{bmatrix}, \cdots
\]
Azimuth and elevation

Knowing \( u \) and also the fact that it corresponds to

\[
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we compute the azimuth:

\[
\phi = \arctan \frac{u_y}{u_x}
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\[
\cdots \text{we compute the azimuth:}
\]
\[
\phi = \arctan \frac{u_y}{u_x}
\]

\[
\text{And the elevation:}
\]
\[
elevation = 90^\circ - \theta = 90^\circ - \arccos u_z
\]
Thank you!