Microphone-Array Signal Processing

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Outline

1. Introduction and Fundamentals
2. Sensor Arrays and Spatial Filtering
3. Optimal Beamforming
4. Adaptive Beamforming
5. DoA Estimation with Microphone Arrays

3. Optimal Beamforming

3.1 Introduction

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- We assume that the appropriate statistics are known.
- Our objective of interest is to estimate the waveform of a plane-wave impinging on the array in the presence of noise and interfering signals.
- Even if a particular beamformer developed in this chapter has good performance, it does not guarantee that its adaptive version (next chapter) will. However, if the performance is poor, it is unlikely that the adaptive version will be useful.

3.2 Optimal Beamformers

MVDR Beamformer

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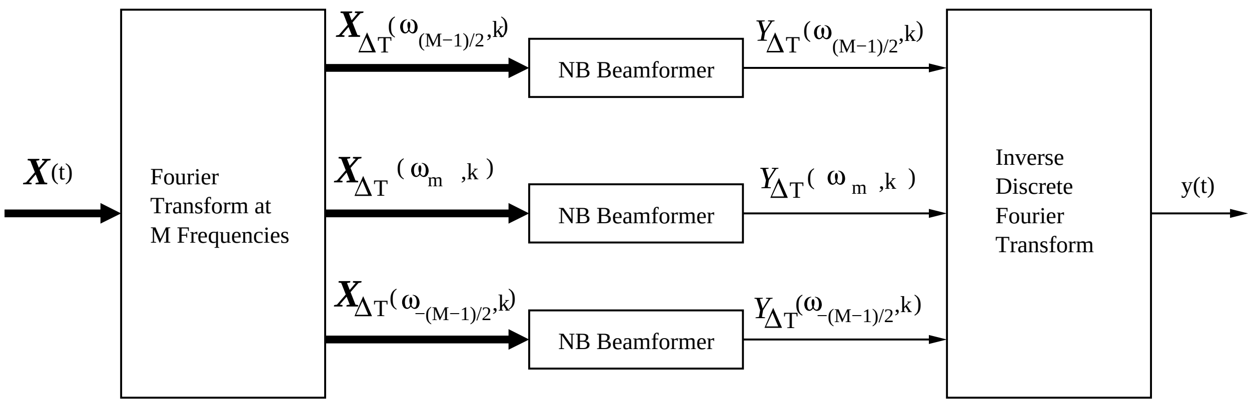

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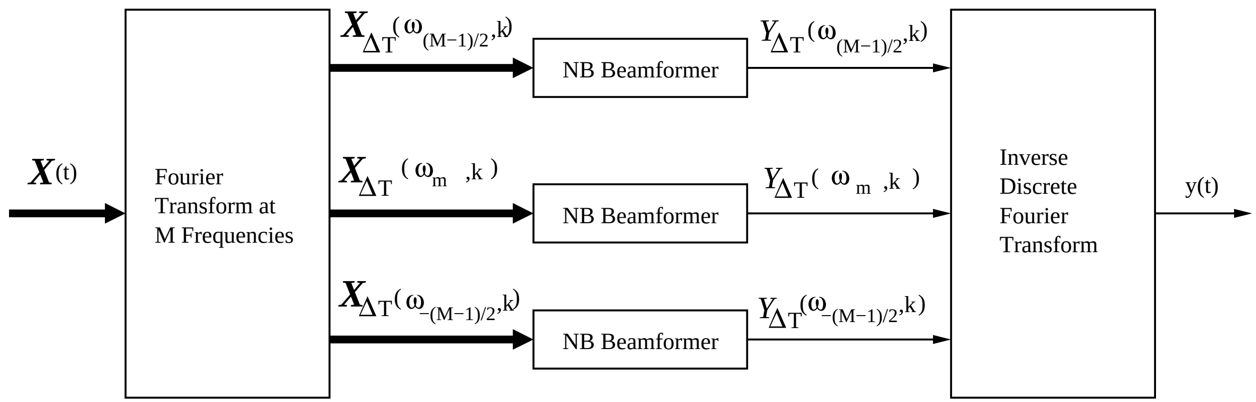

- In order to generate these vectors, divide the observation interval T in K disjoint intervals of duration ΔT: $(k-1)\Delta T \leq t < k\Delta T, k = 1, \cdots, K$.

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- In order to develop the frequency-domain snapshot model for the case in which the desired signals and the interfering signals can de modeled as plane waves, we have two cases: desired signals are deterministic or samples of a random process.

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- Frequency-domain snapshot consists of signal plus noise: $\boldsymbol{X}(\omega) = \boldsymbol{X}_s(\omega) + \boldsymbol{N}(\omega)$
- The signal vector can be written as $\boldsymbol{X}_s(\omega) = F(\omega)\boldsymbol{v}(\omega : \boldsymbol{k}_s)$ where $F(\omega)$ is the frequency-domain snapshot of the source signal and $\boldsymbol{v}(\omega : \boldsymbol{k}_s)$ is the array manifold vector for a plane-wave with wavenumber $\boldsymbol{k}_s$.

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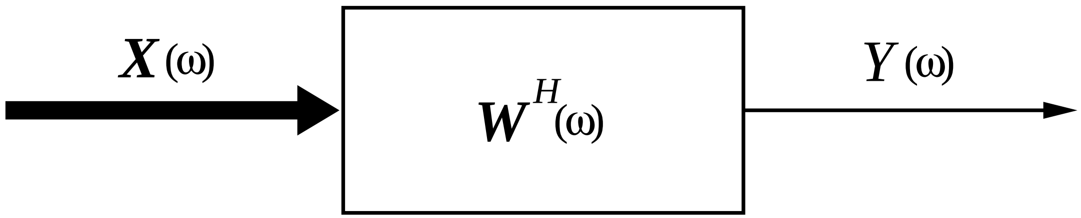
- The noise snapshot is a zero-mean random vector $\boldsymbol{N}(\omega)$ with spectral matrix given by $\boldsymbol{S}_{\boldsymbol{n}}(\omega) = \boldsymbol{S}_{\boldsymbol{c}}(\omega) + \sigma_\omega^2 \boldsymbol{I}$

- We process $\boldsymbol{X}(\omega)$ with the $1 \times N$ operator $\boldsymbol{W}^H(\omega)$:



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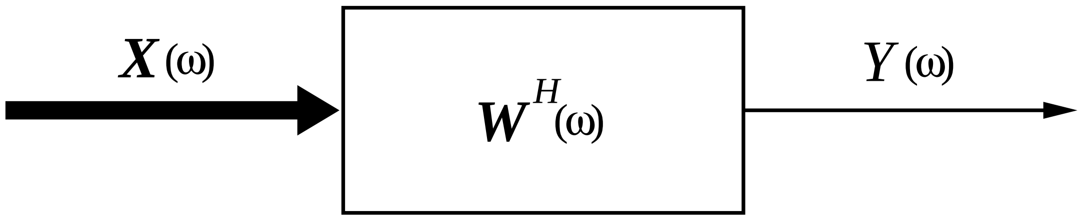

- Distortionless criterion (in the absence of noise):

$$
\begin{aligned}
Y(\omega) &= F(\omega) \\
&= \boldsymbol{W}^H(\omega)\boldsymbol{X}_s(\omega) = F(\omega)\boldsymbol{W}^H(\omega)\boldsymbol{v}(\omega : \boldsymbol{k}_s) \\
\Longrightarrow & \boldsymbol{W}^H(\omega)\boldsymbol{v}(\omega : \boldsymbol{k}_s) = 1
\end{aligned}
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- The mean square of the output noise is:

$$E[|Y_n(\omega)|^2] = \boldsymbol{W}^H(\omega)\boldsymbol{S}_n(\omega)\boldsymbol{W}(\omega)$$

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$$F = \boldsymbol{W}^H(\omega)\boldsymbol{S}_n(\omega)\boldsymbol{W}\omega$$

$$+\lambda\left[\boldsymbol{W}^H(\omega)\boldsymbol{v}(\omega:\boldsymbol{k}_s) - 1\right] + \lambda^*\left[\boldsymbol{v}^H(\omega:\boldsymbol{k}_s)\boldsymbol{W}(\omega) - 1\right]$$

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- ...and the result (suppressing ω and $\boldsymbol{k}_s$) is

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- This result is referred to as MVDR or *Capon* Beamformer.

- The *gradient* of ξ with respect to $\boldsymbol{w}$ (real case):

$$\boldsymbol{\nabla}_{\boldsymbol{w}}\xi = \begin{bmatrix} \frac{\partial \xi}{\partial w_0} \\ \frac{\partial \xi}{\partial w_1} \\ \vdots \\ \frac{\partial \xi}{\partial w_{N-1}} \end{bmatrix}$$

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- which, when $\mathbf{R}$ is symmetric, leads to $\boldsymbol{\nabla}_{\boldsymbol{w}}(\boldsymbol{w}^T\boldsymbol{R}\boldsymbol{w}) = 2\boldsymbol{R}\boldsymbol{w}$

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Constrained Optimal Filtering

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- $\cdots$ which corresponds to $\nabla_{\boldsymbol{w}}\xi = \nabla_{\boldsymbol{a}}\xi + j\nabla_{\boldsymbol{b}}\xi$
- Let us define the *derivative* $\frac{\partial}{\partial \mathbf{w}}$ (with respect to $\mathbf{w}$):

$$\frac{\partial}{\partial \mathbf{w}} = \frac{1}{2}\begin{bmatrix} \frac{\partial}{\partial a_0} - j\frac{\partial}{\partial b_0} \\ \frac{\partial}{\partial a_1} - j\frac{\partial}{\partial b_1} \\ \vdots \\ \frac{\partial}{\partial a_{N-1}} - j\frac{\partial}{\partial b_{N-1}} \end{bmatrix}$$

- The *conjugate derivative* with respect to $\mathbf{w}$ is

$$\frac{\partial}{\partial \mathbf{w}^*} = \frac{1}{2} \begin{bmatrix} \frac{\partial}{\partial a_0} + j\frac{\partial}{\partial b_0} \\ \frac{\partial}{\partial a_1} + j\frac{\partial}{\partial b_1} \\ \vdots \\ \frac{\partial}{\partial a_{N-1}} + j\frac{\partial}{\partial b_{N-1}} \end{bmatrix}$$

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- $\boldsymbol{\nabla}_{\mathbf{w}} E[e(k)e^*(k)] = E\{e^*(k)[\boldsymbol{\nabla}_{\mathbf{w}} e(k)] + e(k)[\boldsymbol{\nabla}_{\mathbf{w}} e^*(k)]\}$

- We compute each gradient ...

$$\begin{aligned}\boldsymbol{\nabla}_{\mathbf{w}} e(k) =& \boldsymbol{\nabla}_{\mathbf{a}}[d(k) - \mathbf{w}^H \mathbf{x}(k)] + j\boldsymbol{\nabla}_{\mathbf{b}}[d(k) - \mathbf{w}^H \mathbf{x}(k)] \\ =& - \mathbf{x}(k) - \mathbf{x}(k) = -2\mathbf{x}(k)\end{aligned}$$

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Constrained Optimal Filtering

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- such that the final result is

$$\begin{aligned}\nabla_{\mathbf{w}} E[e(k) e^*(k)] =& -2E[e^*(k)\mathbf{x}(k)] \\ =& -2E[\mathbf{x}(k)[d(k) - \mathbf{w}^H \mathbf{x}(k)]^*\} \\ =& -2\underbrace{E[\mathbf{x}(k) d^*(k)]}_{\mathbf{p}} + 2\underbrace{E[\mathbf{x}(k)\mathbf{x}^H(k)]}_{\mathbf{R}} \mathbf{w}\end{aligned}$$

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- The output of the processor is $y(k) = \mathbf{w}^H\mathbf{x}(k)$.
- It is worth mentioning that the most general case corresponds to having a reference signal, $d(k)$. It is, however, usual to have no reference signal as in Linearly-Constrained Minimum-Variance (LCMV) applications. In LCMV, if $\mathbf{f} = 1$, the system is often referred to as Minimum-Variance Distortionless Response (MVDR).

- Using Lagrange multipliers, we form

$$\xi(k) = E[e(k)e^*(k)] + \boldsymbol{\mathcal{L}}_R^T Re[\mathbf{C}^H\mathbf{w} - \mathbf{f}] + \boldsymbol{\mathcal{L}}_I^T Im[\mathbf{C}^H\mathbf{w} - \mathbf{f}]$$

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$$\begin{aligned}
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&= E[e(k)e^*(k)] + \frac{1}{2}\boldsymbol{\mathcal{L}}^H(\mathbf{C}^H\mathbf{w} - \mathbf{f}) + \frac{1}{2}\boldsymbol{\mathcal{L}}^T(\mathbf{C}^T\mathbf{w}^* - \mathbf{f}^*)
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- Noting that $e(k) = d(k) - \mathbf{w}^H\mathbf{x}(k)$, we compute:

$$\begin{aligned}\boldsymbol{\nabla}_{\mathbf{w}}\xi(k) &= \boldsymbol{\nabla}_{\mathbf{w}}\left\{E[e(k)e^*(k)] + \frac{1}{2}\boldsymbol{\mathcal{L}}^H(\mathbf{C}^H\mathbf{w} - \mathbf{f}) + \frac{1}{2}\boldsymbol{\mathcal{L}}^T(\mathbf{C}^T\mathbf{w}^* - \mathbf{f}^*)\right\} \\ &= E[-2\mathbf{x}(k)e^*(k)] + \mathbf{0} + \mathbf{C}\boldsymbol{\mathcal{L}} \\ &= -2E[\mathbf{x}(k)d^*(k)] + 2E[\mathbf{x}(k)\mathbf{x}^H(k)]\mathbf{w} + \mathbf{C}\boldsymbol{\mathcal{L}}\end{aligned}$$

Constrained Optimal Filtering

- By using $\mathbf{R} = E[\mathbf{x}(k)\mathbf{x}^H(k)]$ and $\mathbf{p} = E[d^*(k)\mathbf{x}(k)]$, the gradient is equated to zero and the results can be written as (note that stationarity was assumed for the input and reference signals): $-2\mathbf{p} + 2\mathbf{R}\mathbf{w} + \mathbf{C}\boldsymbol{\mathcal{L}} = \mathbf{0}$

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- By replacing $\boldsymbol{\mathcal{L}}$, we obtain the *Wiener solution* for the linearly constrained adaptive filter:
$\mathbf{w}_{opt} = \mathbf{R}^{-1}\mathbf{p} + \mathbf{R}^{-1}\mathbf{C}(\mathbf{C}^H\mathbf{R}^{-1}\mathbf{C})^{-1}(\mathbf{f} - \mathbf{C}^H\mathbf{R}^{-1}\mathbf{p})$

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- Note that if $d(k) = 0$, then $\mathbf{p} = \mathbf{0}$, and we have (LCMV):
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- Also note that in case we do not have constraints ($\mathbf{C}$ and $\mathbf{f}$ are nulls), the optimal solution above becomes the *unconstrained* Wiener solution $\mathbf{R}^{-1}\mathbf{p}$.

We start by doing a transformation in the coefficient vector.

- Let $\mathbf{T} = [\mathbf{C}\ \mathbf{B}]$ such that

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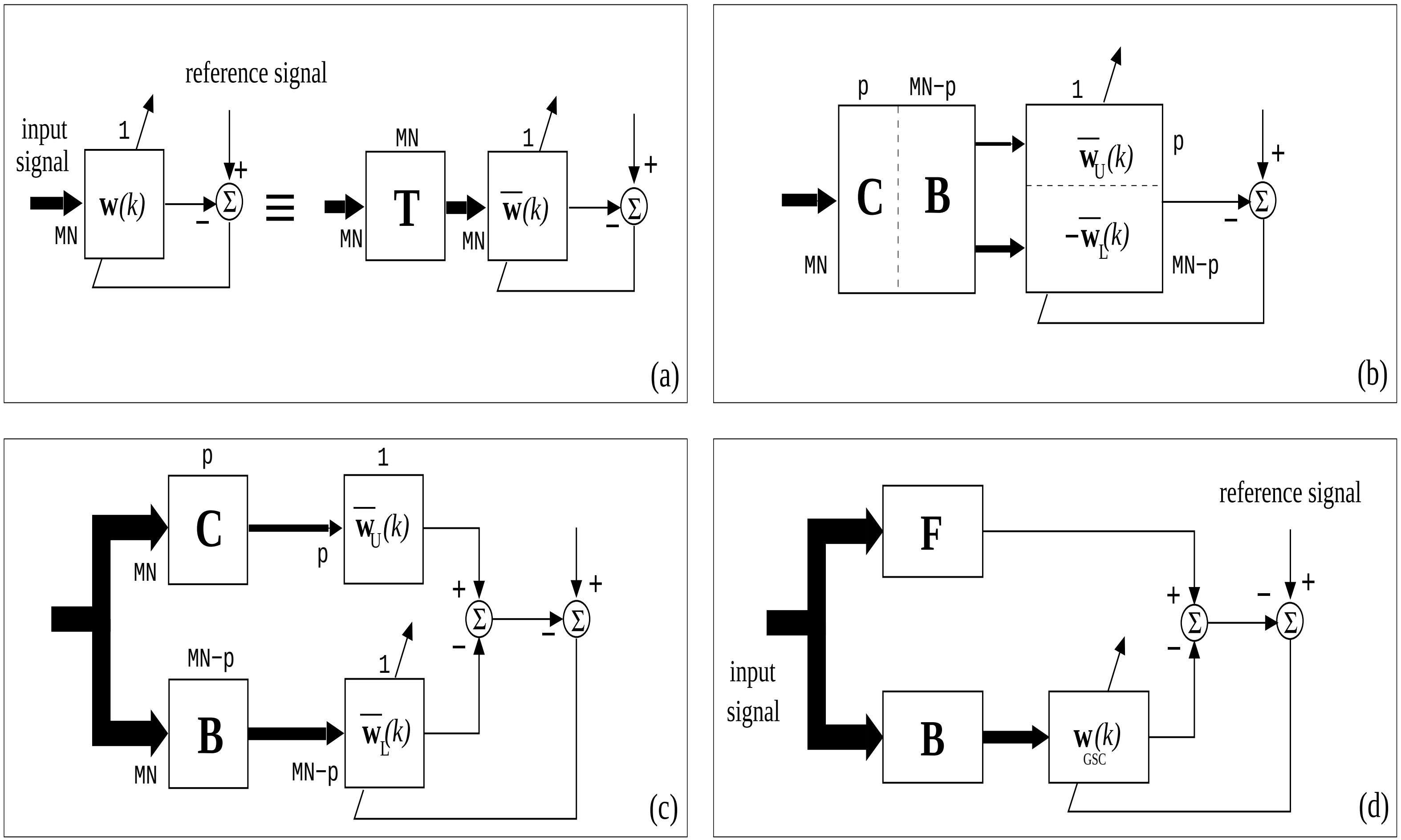
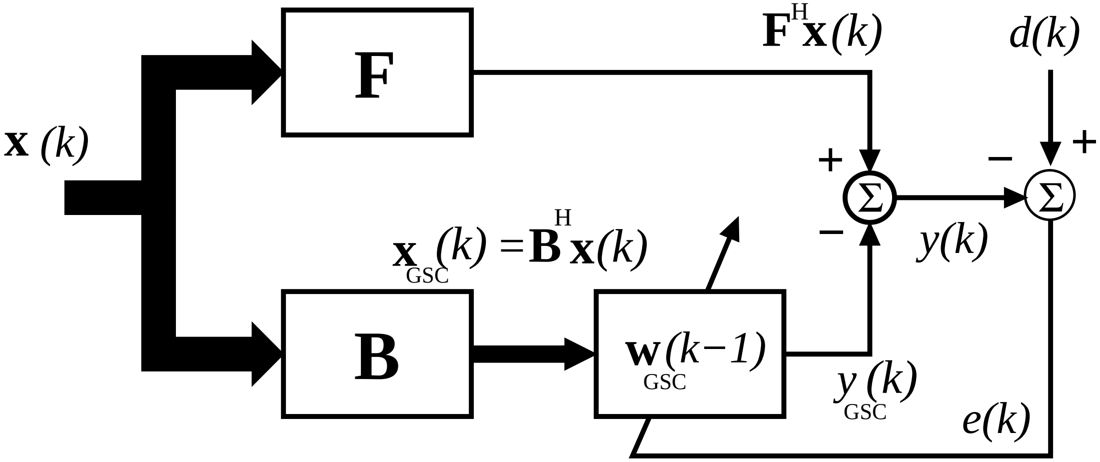
- $\bar{\mathbf{w}}_U$ is fixed and termed the quiescent weight vector; the minimization process will be carried out only in the lower part, also designated $\mathbf{w}_{GSC} = \bar{\mathbf{w}}_L$.

- It is shown below how to split the transformation matrix into two parts: a fixed path and an *adaptive* path.



- This structure (detailed below) was named the Generalized Sidelobe Canceller (GSC).



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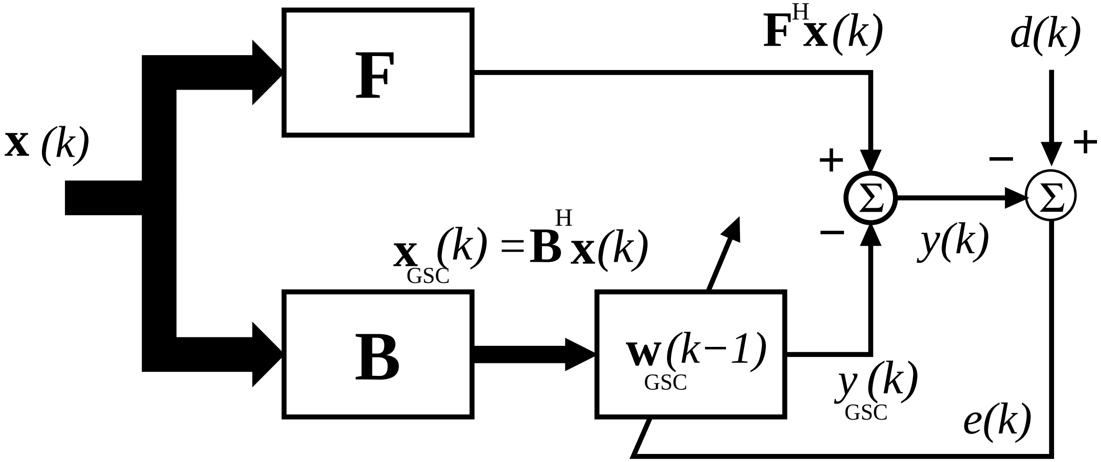

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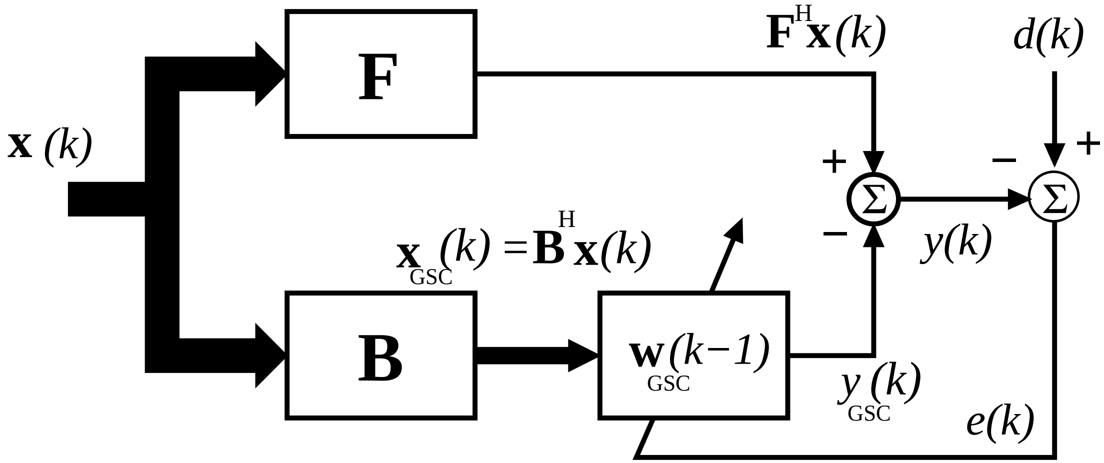

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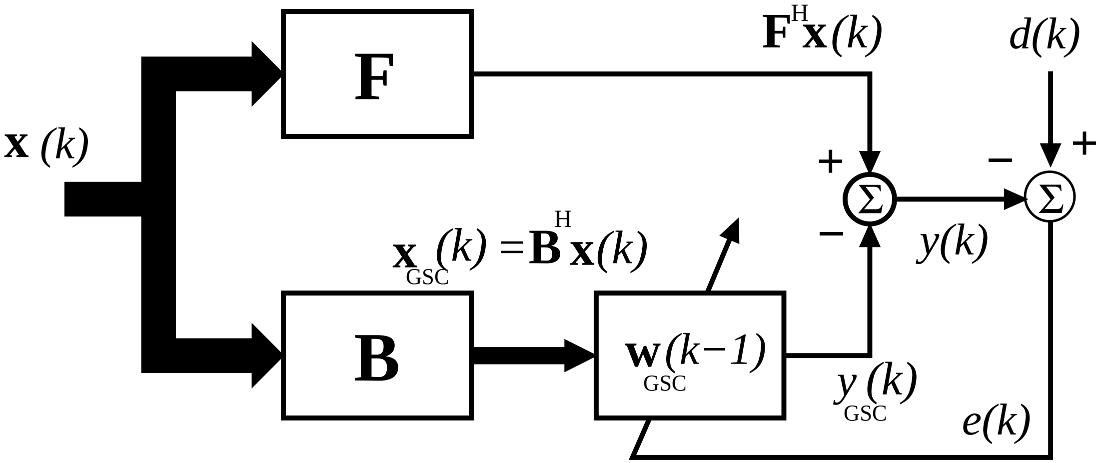

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- Knowing that $\mathbf{T} = [\mathbf{C}\ \mathbf{B}]$ and that $\mathbf{T}^H\mathbf{T} = \boldsymbol{I}$, it follows that $\boldsymbol{P} = \mathbf{I} - \mathbf{C}(\mathbf{C}^H\mathbf{C})^{-1}\mathbf{C}^H = \mathbf{B}(\mathbf{B}^H\mathbf{B})\mathbf{B}^H$.

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\begin{aligned}
\mathbf{p}_{GSC} &= E[d_{GSC}^*\mathbf{x}_{GSC}] \\
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&= E[-\mathbf{B}^H d^*\mathbf{x} + \mathbf{B}^H\mathbf{x}\mathbf{x}^H\mathbf{F}] \\
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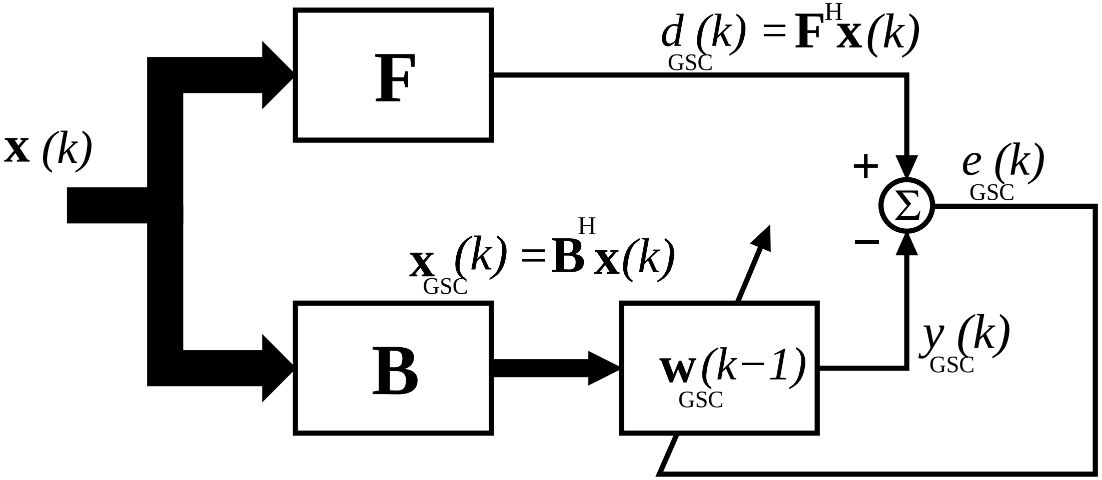
- A common case is when $d(k) = 0$:



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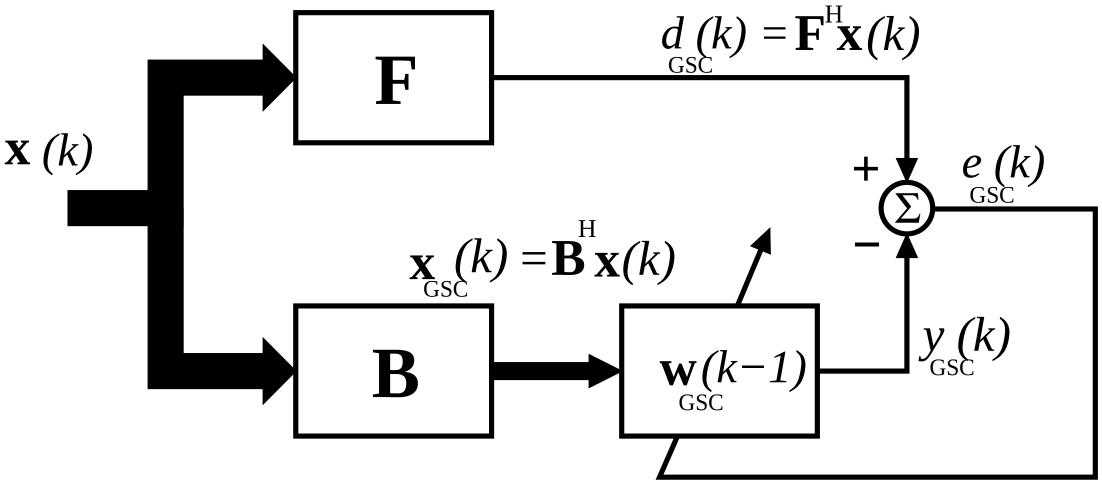

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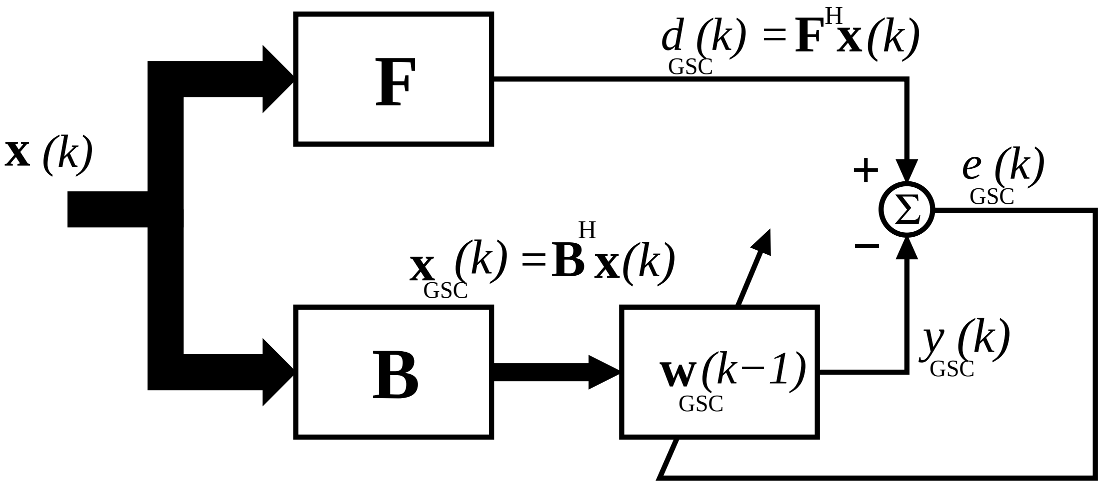

- We have dropped the negative sign that should exist according to the notation used. Although we define $e_{GSC}(k) = -e(k)$, the inversion of the sign in the error signal actually results in the same results because the error function is always based on the absolute value.
- In this case, the optimum filter $\mathbf{w}_{OPT}$ is: $\mathbf{F} - \mathbf{B}\mathbf{w}_{GSC-OPT} = \mathbf{F} - \mathbf{B}(\mathbf{B}^H\mathbf{R}\mathbf{B})^{-1}\mathbf{B}^H\mathbf{R}\mathbf{F} = \mathbf{R}^{-1}\mathbf{C}(\mathbf{C}^H\mathbf{R}^{-1}\mathbf{C})^{-1}\mathbf{f}$ (LCMV solution)

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- Let us recall the paper by Griffiths and Jim where the term GSC was coined; let

$$\mathbf{C}^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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- With simple constraint matrices, simple blocking matrices satisfying $\mathbf{B}^T\mathbf{C} = \mathbf{0}$ are possible.

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$\mathbf{B}_3^T$ is given by:

$$\begin{bmatrix} -0.50 & -0.17 & -0.17 & 0.83 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.25 & -0.42 & 0.08 & 0.08 & 0.75 & -0.25 & -0.25 & -0.25 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.25 & -0.42 & 0.08 & 0.08 & -0.25 & 0.75 & -0.25 & -0.25 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.25 & -0.42 & 0.08 & 0.08 & -0.25 & -0.25 & 0.75 & -0.20 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.25 & -0.42 & 0.08 & 0.08 & -0.25 & -0.25 & -0.25 & 0.75 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.25 & 0.08 & -0.42 & 0.08 & 0.00 & 0.00 & 0.00 & 0.00 & 0.75 & -0.25 & -0.25 & -0.25 \\ 0.25 & 0.08 & -0.42 & 0.08 & 0.00 & 0.00 & 0.00 & 0.00 & -0.25 & 0.75 & -0.25 & -0.25 \\ 0.25 & 0.08 & -0.42 & 0.08 & 0.00 & 0.00 & 0.00 & 0.00 & -0.25 & -0.25 & 0.75 & -0.25 \\ 0.25 & 0.08 & -0.42 & 0.08 & 0.00 & 0.00 & 0.00 & 0.00 & -0.25 & -0.25 & -0.25 & 0.75 \end{bmatrix}$$

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- Finally, a new efficient linearly constrained adaptive scheme which can also be visualized as a GSC structure can be found in [Campos&Werner&Apolinário IEEE-TSP Sept. 2002].