Microphone-Array Signal Processing

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IME – Lab. Processamento Digital de Sinais

Outline

1. Introduction and Fundamentals
2. Sensor Arrays and Spatial Filtering
3. Optimal Beamforming
4. Adaptive Beamforming
5. DoA Estimation with Microphone Arrays

2. Sensor Arrays and Spatial Filtering

2.1 Wavenumber-Frequency Space

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spatial frequency: wavenumber

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$$s(\mathbf{x}, t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\mathbf{k}, \omega) e^{j(\omega t - \mathbf{k}^T \mathbf{x})} d\mathbf{k}\, d\omega$$

Space-time Fourier Transform

We have already concluded that if the space-time signal is a propagating waveform such that $s(\mathbf{x},t) = s(t - \boldsymbol{\alpha}_0^T\mathbf{x})$, then its Fourier transform is equal to

$$S(\mathbf{k},\omega) = S(\omega)\delta(\mathbf{k} - \omega\boldsymbol{\alpha}_0)$$

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This means that $s(\mathbf{x}, t)$ only has energy along the direction of $\mathbf{k} = \mathbf{k}_0 = \omega\boldsymbol{\alpha}_0$ in the wavenumber-frequency space.

2.2 Frequency-Wavenumber (WN) Response and Beam patterns (BP)

Signals at the sensors

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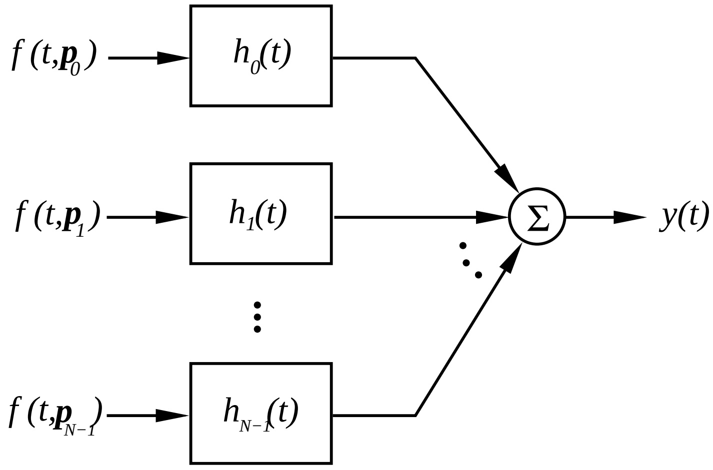
$$\boldsymbol{f}(t, \boldsymbol{p}) = \begin{bmatrix} f(t, \boldsymbol{p}_0) \\ f(t, \boldsymbol{p}_1) \\ \vdots \\ f(t, \boldsymbol{p}_{N-1}) \end{bmatrix}$$

Array output



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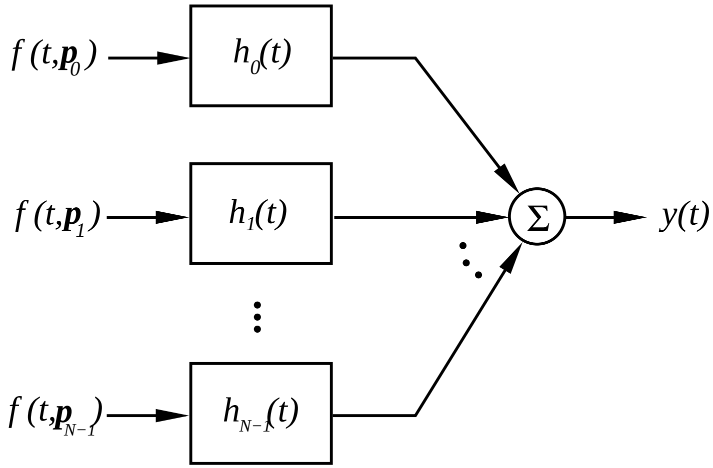

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} h_n(t-\tau) f_n(\tau, \boldsymbol{p}_n) d\tau \\
&= \int_{-\infty}^{\infty} \boldsymbol{h}^T(t-\tau) \boldsymbol{f}(\tau, \boldsymbol{p}) d\tau
\end{aligned}
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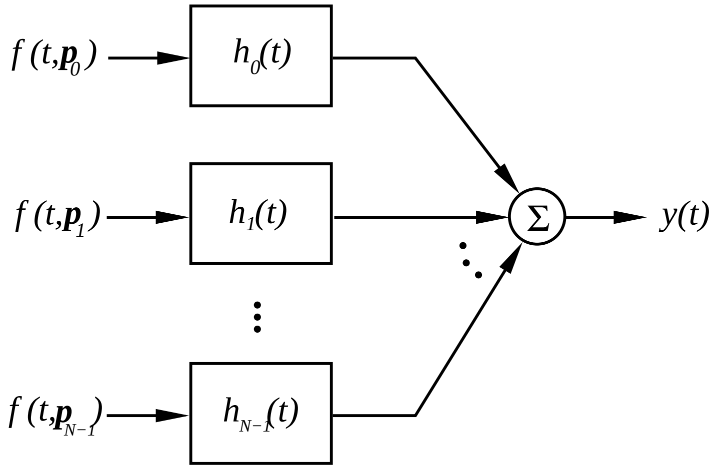

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where $\boldsymbol{h}(t) = [h_o(t) \;\; h_1(t) \;\; \cdots \;\; h_{N-1}(t)]^T$

In the frequency domain, $\cdots$

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$$\boldsymbol{H}(\omega) = \int_{-\infty}^{\infty} \boldsymbol{h}(t) e^{-j\omega t} dt$$

$$\boldsymbol{F}(\omega) = \boldsymbol{F}(\omega, \boldsymbol{p}) = \int_{-\infty}^{\infty} \boldsymbol{f}(t, \boldsymbol{p}) e^{-j\omega t} dt$$

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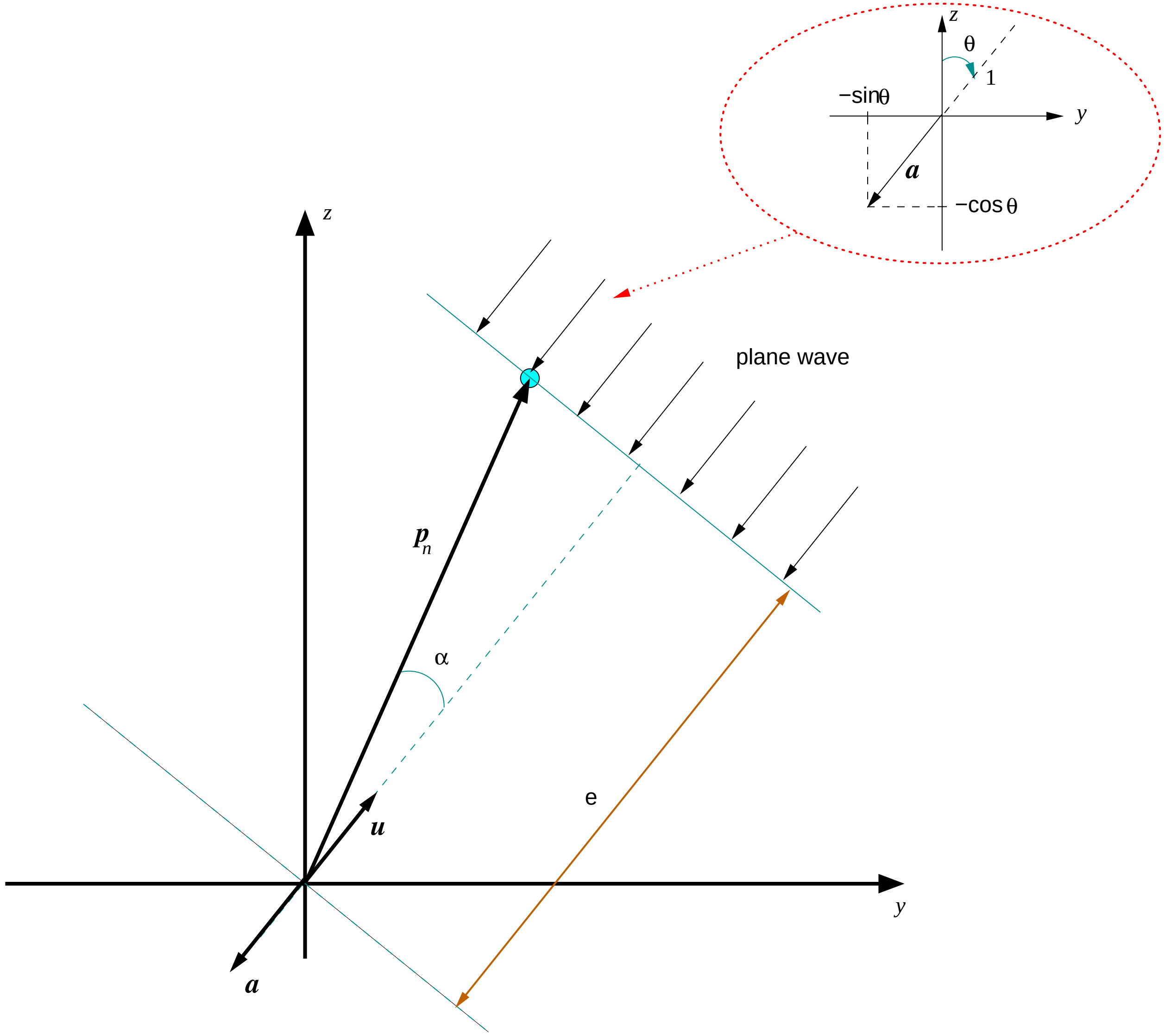
- If $f(t)$ is the signal that would be received at the origin, then:

$$\boldsymbol{f}(t, \boldsymbol{p}) = \begin{bmatrix} f(t-\tau_0) \\ f(t-\tau_1) \\ \vdots \\ f(t-\tau_{N-1}) \end{bmatrix}$$

Plane wave (assuming $\phi = 90^o$)



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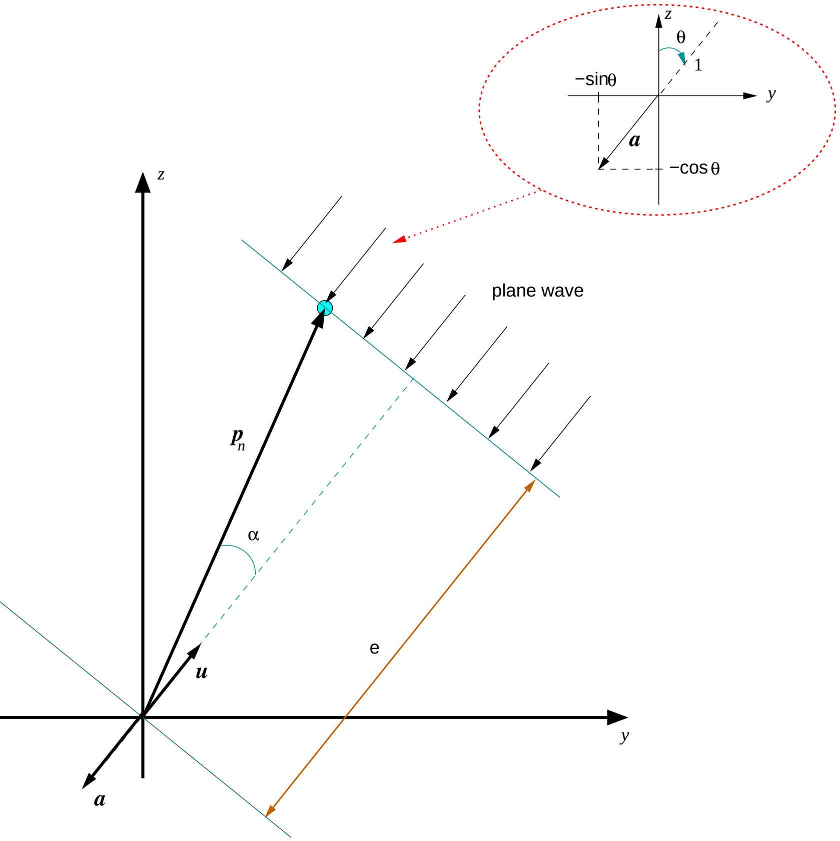

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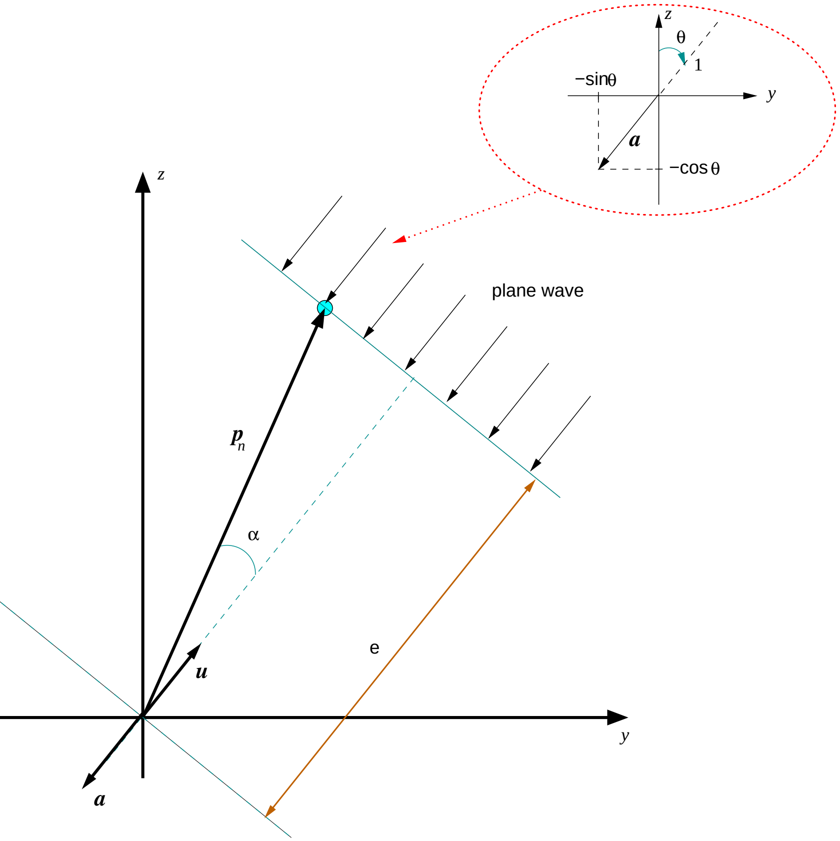

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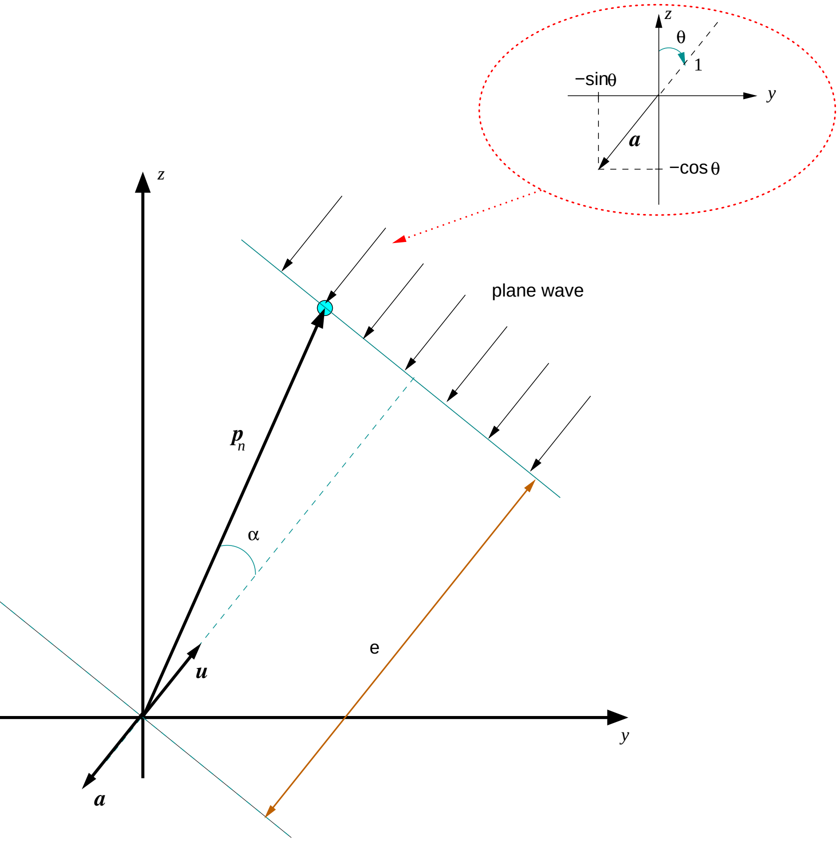

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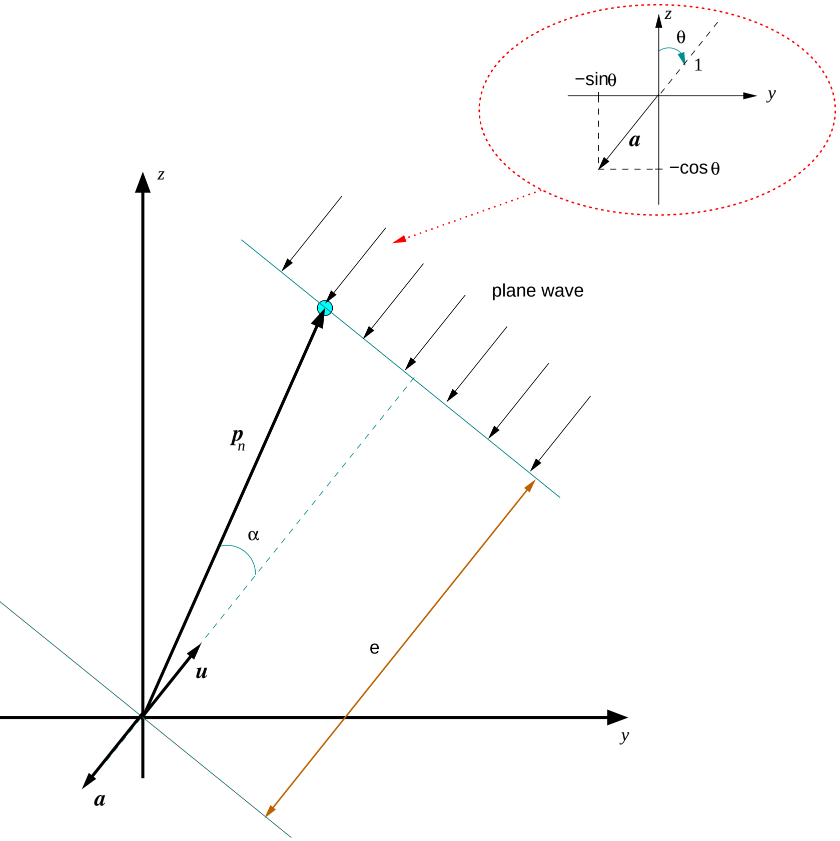

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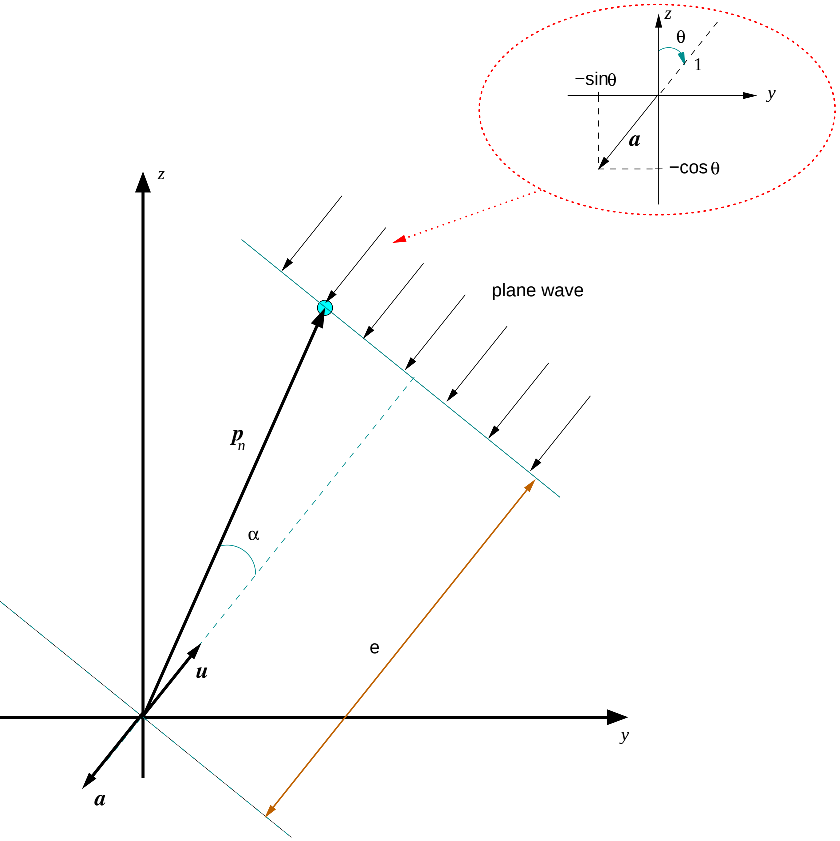

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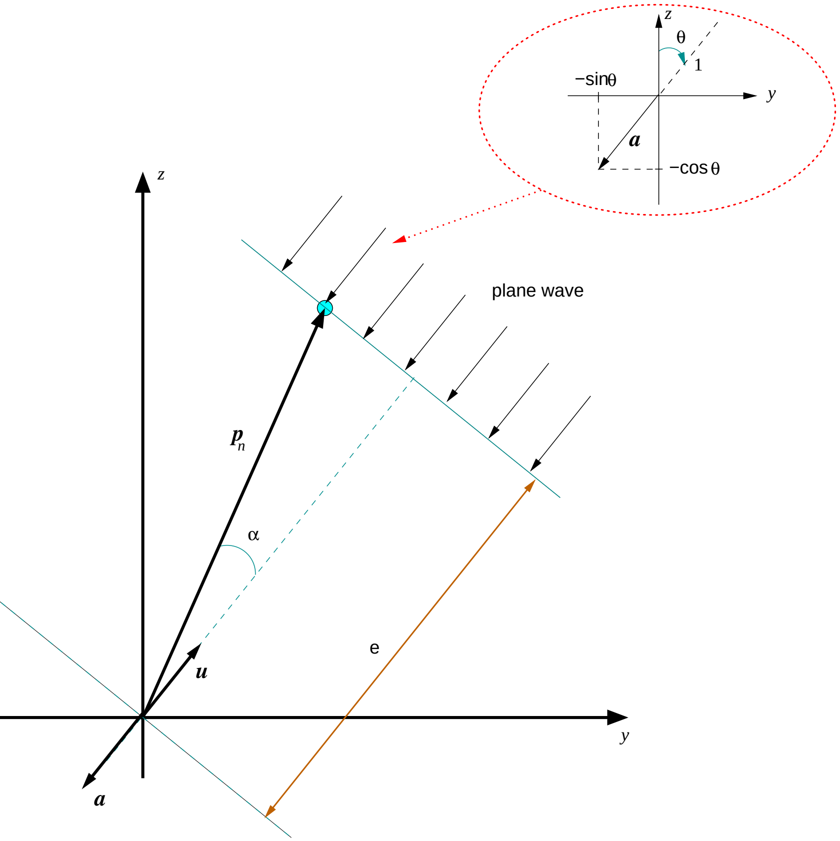

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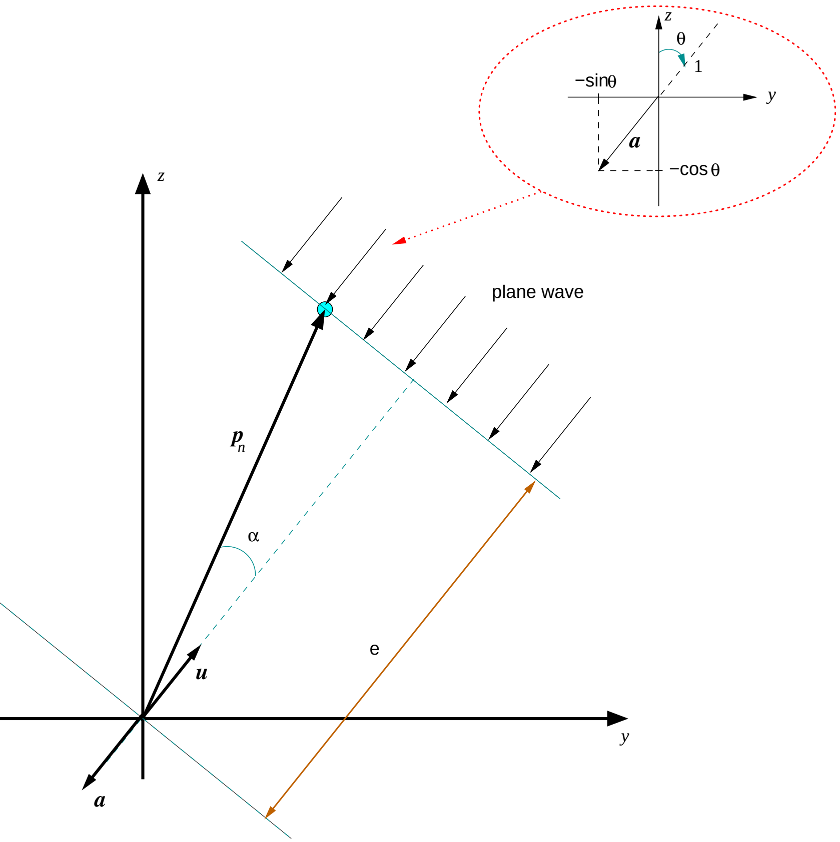

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$_n$ is the time since the plane wave hits the sensor at location $\mathbf{p}_n$ until it reaches point $(0, 0)$.

- Then, we have:

$$\boldsymbol{F}(\omega) = \begin{bmatrix} \int_{-\infty}^{\infty} e^{-j\omega t} f(t-\tau_0) dt \\ \int_{-\infty}^{\infty} e^{-j\omega t} f(t-\tau_1) dt \\ \vdots \\ \int_{-\infty}^{\infty} e^{-j\omega t} f(t-\tau_{N-1}) dt \end{bmatrix}$$

$$= \begin{bmatrix} e^{-j\omega\tau_0} \\ e^{-j\omega\tau_1} \\ \vdots \\ e^{-j\omega\tau_{N-1}} \end{bmatrix} F(\omega)$$

Definition of Wavenumber

- For plane waves propagating in a locally homogeneous medium:

$$\boldsymbol{k} = \frac{\omega}{c}\boldsymbol{a} = \frac{2\pi}{c/f}\boldsymbol{a} = \frac{2\pi}{\lambda}\boldsymbol{a} = -\frac{2\pi}{\lambda}\boldsymbol{u}$$

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- Note that $|\boldsymbol{k}| = \frac{2\pi}{\lambda}$
- Therefore

$$\omega\tau_n = \frac{\omega}{c}\boldsymbol{a}^T\boldsymbol{p}_n = \boldsymbol{k}^T\boldsymbol{p}_n$$

- And we have

$$\boldsymbol{F}(\omega) = \begin{bmatrix} e^{-j\boldsymbol{k}^T\boldsymbol{p}_0} \\ e^{-j\boldsymbol{k}^T\boldsymbol{p}_1} \\ \vdots \\ e^{-j\boldsymbol{k}^T\boldsymbol{p}_{N-1}} \end{bmatrix} F(\omega) = F(\omega)\boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k})$$

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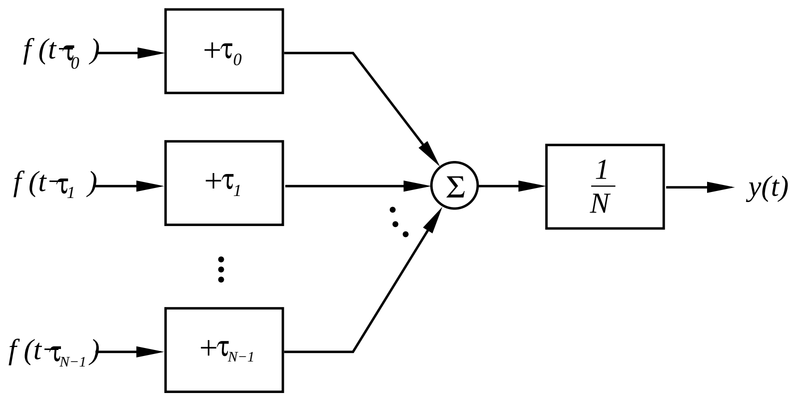
- In this particular example, we can use $h_n(t) = \frac{1}{N}\delta(t + \tau_n)$ such that

$$y(t) = f(t)$$

Following, we have the delay-and-sum beamformer.

Delay-and-sum Beamformer



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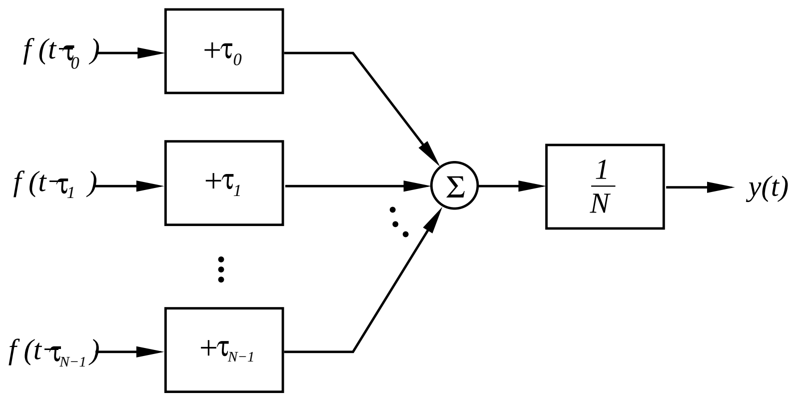

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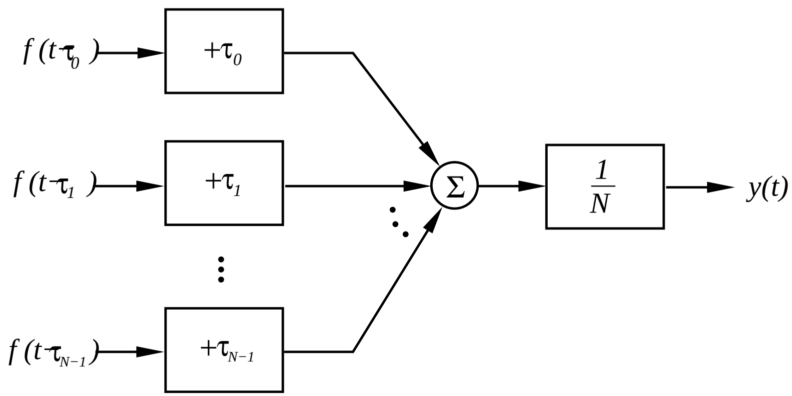

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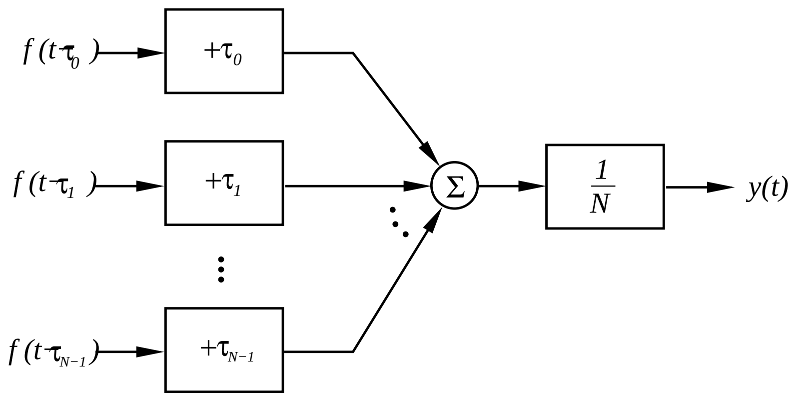

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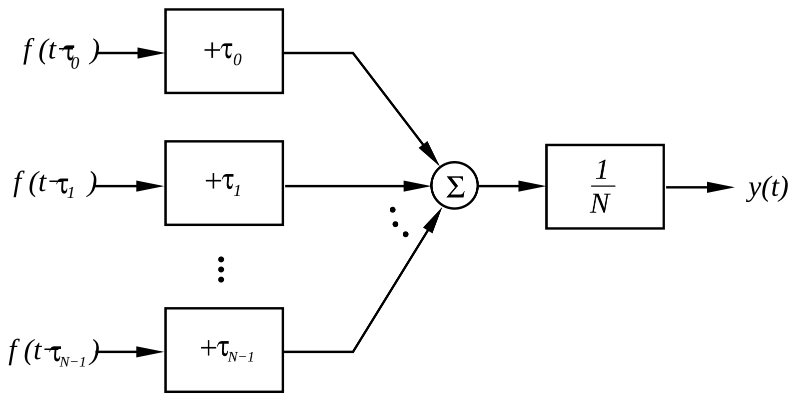

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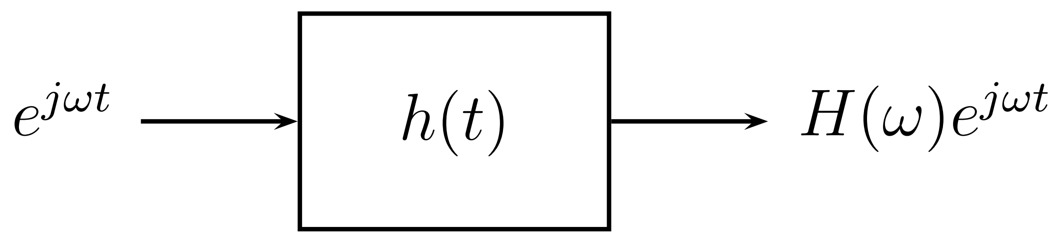
LTI System



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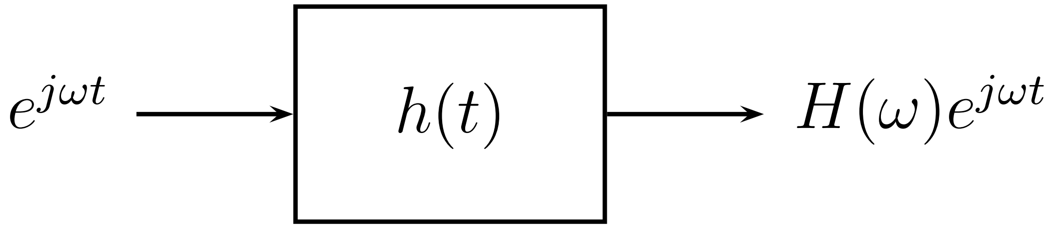
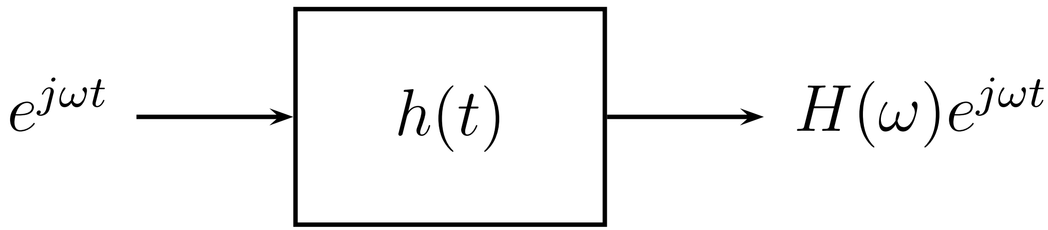


- Space-time signals (base functions):

$$f_n(t,\boldsymbol{p}) = e^{j\omega(t-\tau_n)} = e^{j(\omega t - \boldsymbol{k}^T \boldsymbol{p}_n)}$$

Note that $\omega\tau_n = \boldsymbol{k}^T \boldsymbol{p}_n$



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- $\therefore\ \boldsymbol{f}(t,\boldsymbol{p}) = e^{j\omega t}\boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k})$

Frequency-Wavenumber Response Function

- The response of the array to this plane wave is:

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- And we define the Frequency-Wavenumber Response Function:

Upsilon

$$\Upsilon(\omega, \boldsymbol{k}) \triangleq \boldsymbol{H}^T(\omega)\boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k})$$

$\Upsilon(\omega, \boldsymbol{k})$ describes the complex gain of an array to an input plane wave with wavenumber $\boldsymbol{k}$ and temporal frequency ω.

Beam Pattern and Bandpass Signal

- BEAM PATTERN is the Frequency Wavenumber Response Function evaluated versus the direction:

$$B(\omega : \theta, \phi) = \Upsilon(\omega, \boldsymbol{k})$$

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$$f(t, \boldsymbol{p}_n) = \sqrt{2}Re\{\tilde{f}(t, \boldsymbol{p}_n)e^{j\omega_c t}\}, n = 0, 1, \cdots, N-1$$

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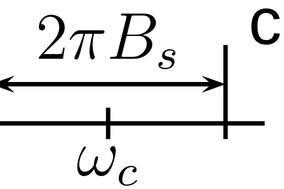
- ω_c corresponds to the carrier frequency and the complex envelope $\tilde{f}(t, \boldsymbol{p}_n)$ is bandlimited to the region



$$|\underbrace{\omega - \omega_c}_{\omega_L}| \leq 2\pi B_s/2$$

Bandlimited and Narrowband Signals

- Bandlimited plane wave:

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- In Narrowband (NB) signals, $B_s \Delta T_{max} \ll 1$
  $\Rightarrow \tilde{f}(t - \tau_n) \simeq \tilde{f}(t)$ and
  $f(t, \boldsymbol{p}_n) = \sqrt{2} Re\{\tilde{f}(t) e^{-j\omega_c \tau_n} e^{j\omega_c t}\}$

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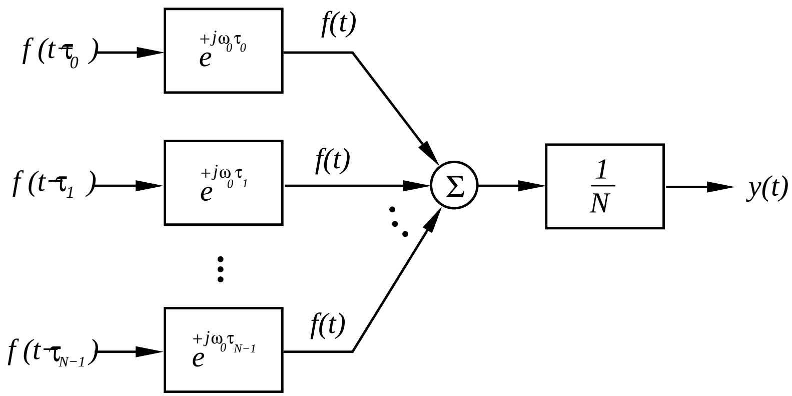

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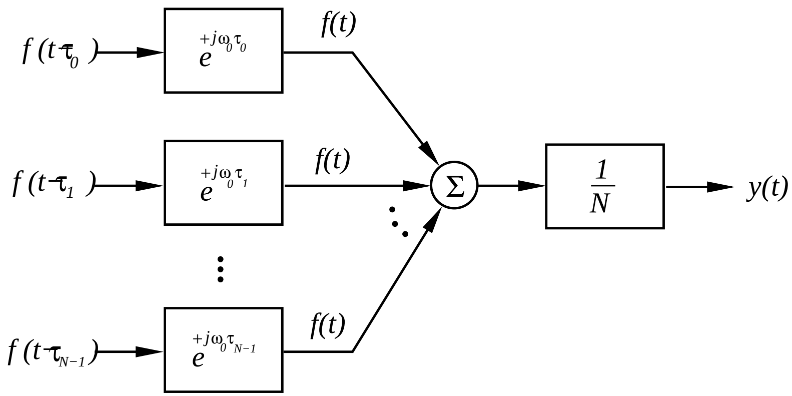

- The phased array can be implemented adjusting the gain and phase to achieve a desired beam pattern

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NB Beamformers

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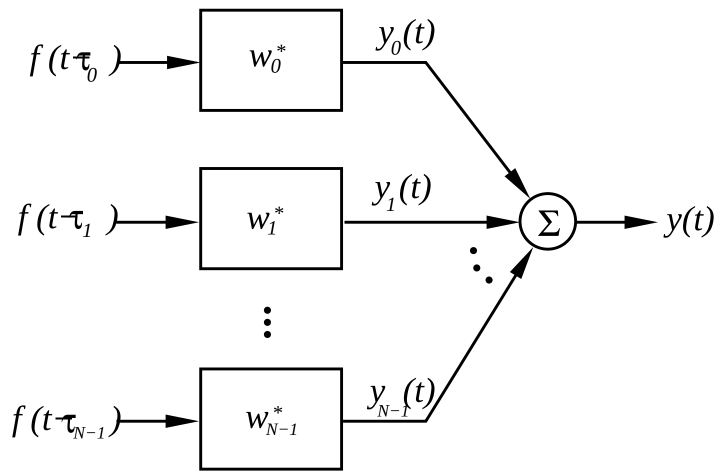
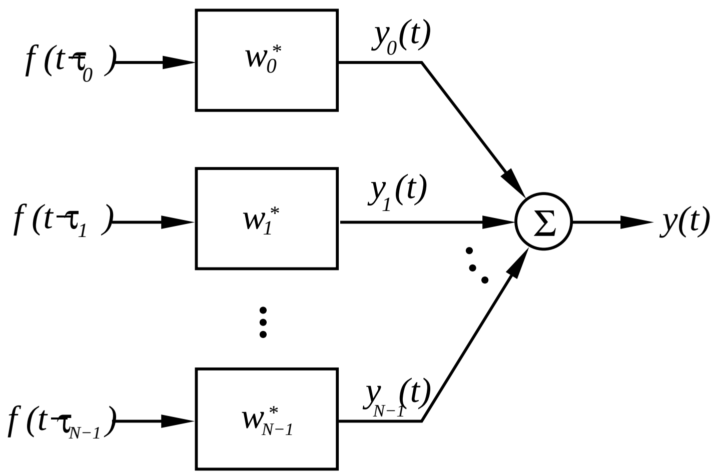


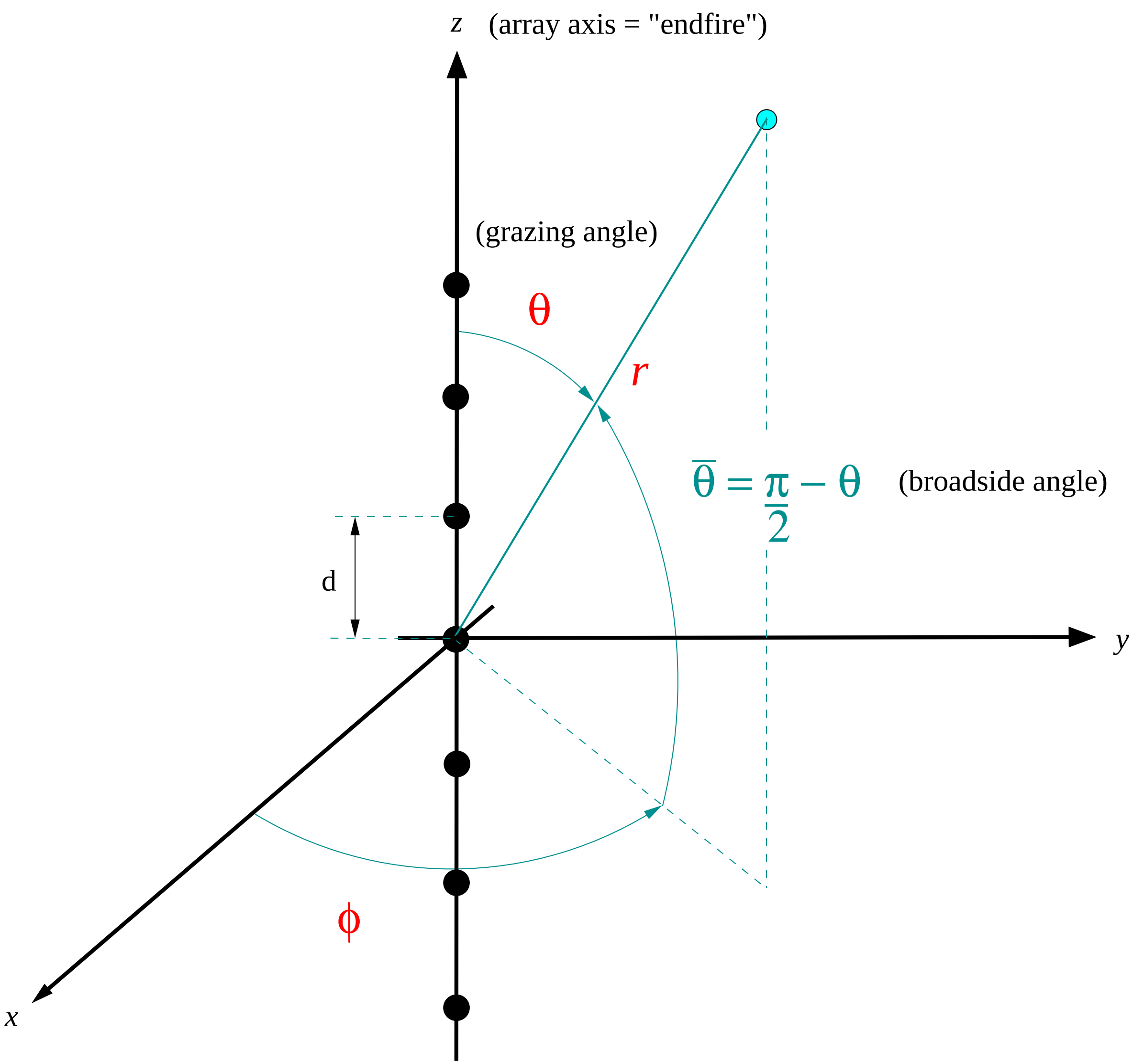
- In narrowband beamformers: $y(t, \boldsymbol{k}) = \boldsymbol{w}^H \boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k}) e^{j\omega t}$



- $\Upsilon(\omega, \boldsymbol{k}) = \underbrace{\boldsymbol{w}^H}_{\boldsymbol{H}^T(\omega)} \boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k})$

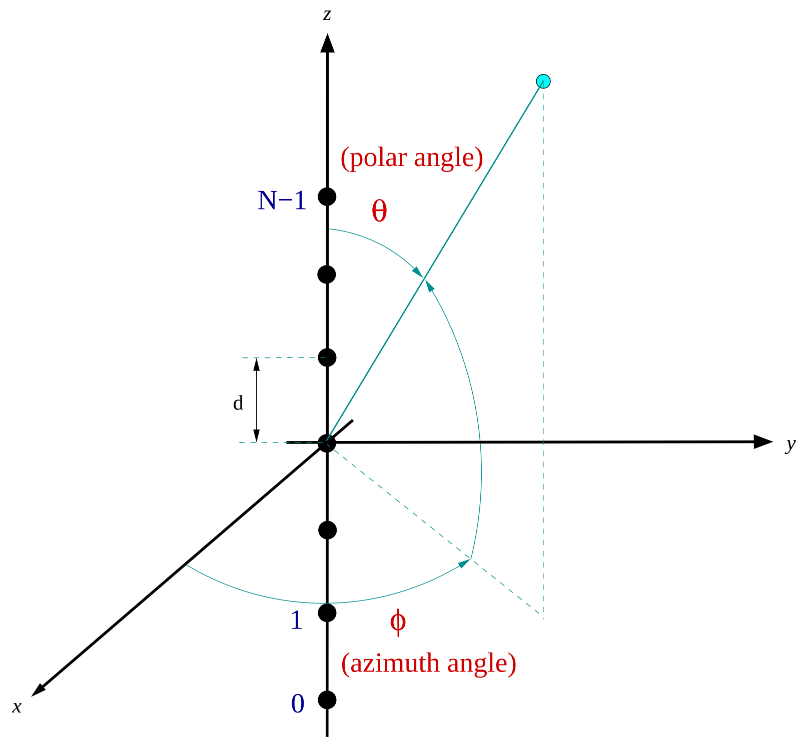
2.3 Uniform Linear Arrays (ULA)

Uniformly Spaced Linear Arrays



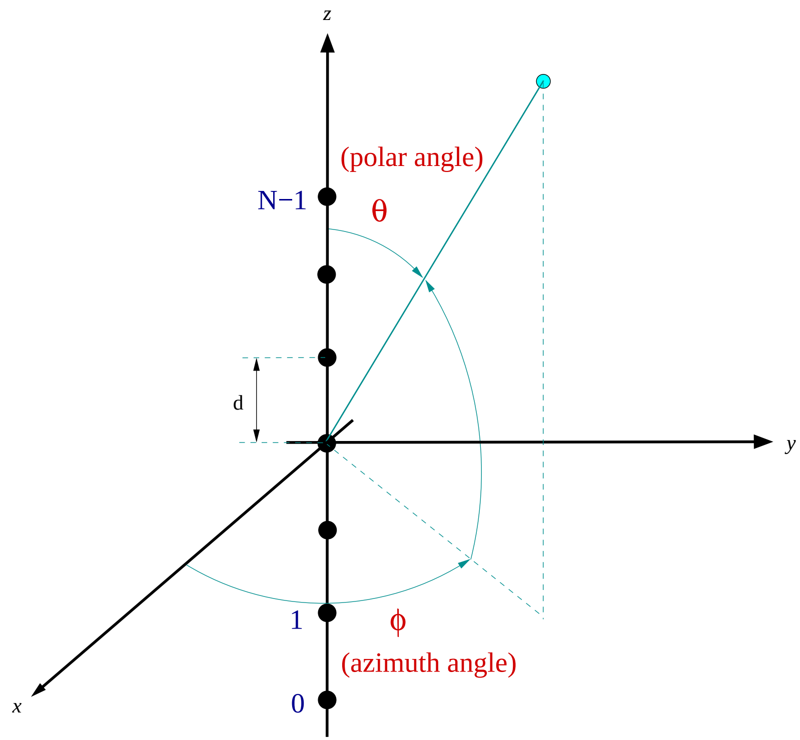
ULA

- An ULA along axis z:



ULA

- An ULA along axis z:

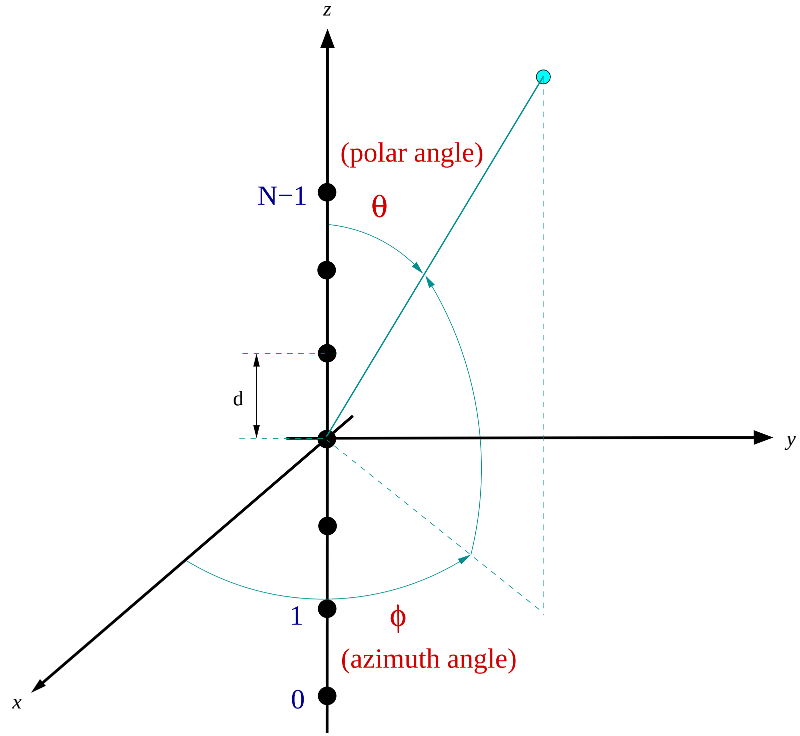


- Location of the elements:

$$\begin{cases} p_{zn} = (n - \frac{N-1}{2})d, \text{ for } n = 0, 1, \cdots, N-1 \\ p_{xn} = p_{yn} = 0 \end{cases}$$

ULA

- An ULA along axis z:



- Location of the elements:

$$\begin{cases} p_{zn} = (n - \frac{N-1}{2})d, \text{ for } n = 0, 1, \cdots, N-1 \\ p_{xn} = p_{yn} = 0 \end{cases}$$

- Therefore, $\boldsymbol{p}_n = \begin{bmatrix} 0 \\ 0 \\ (n - \frac{N-1}{2})d \end{bmatrix}$

- Array manifold vector:

$$\boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k}) = \begin{bmatrix} e^{-j\boldsymbol{k}^T\boldsymbol{p}_0} \\ \vdots \\ e^{-j\boldsymbol{k}^T\boldsymbol{p}_n} \\ \vdots \\ e^{-j\boldsymbol{k}^T\boldsymbol{p}_{N-1}} \end{bmatrix}$$

$$\begin{bmatrix} k_x & k_y & k_z \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \left[n - \frac{N-1}{2}\right] d \end{bmatrix}$$

- Array manifold vector:

$$\boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k}) = \left[e^{-j\boldsymbol{k}^T\boldsymbol{p}_0} \quad e^{-j\boldsymbol{k}^T\boldsymbol{p}_1} \quad \ldots \quad e^{-j\boldsymbol{k}^T\boldsymbol{p}_{N-1}} \right]^T$$

$$\therefore \boldsymbol{v}_{\boldsymbol{k}}(\boldsymbol{k}) = \boldsymbol{v}_{\boldsymbol{k}}(k_z) = \begin{bmatrix} e^{+j\frac{(N-1)}{2}k_z d} \\ e^{+j(\frac{N-1}{2}-1)k_z d} \\ \vdots \\ e^{-j(\frac{N-1}{2})k_z d} \end{bmatrix}$$