Microphone-Array Signal Processing

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IME – Lab. Processamento Digital de Sinais

Outline

1. Introduction and Fundamentals
2. Sensor Arrays and Spatial Filtering
3. Optimal Beamforming
4. Adaptive Beamforming
5. DoA Estimation with Microphone Arrays

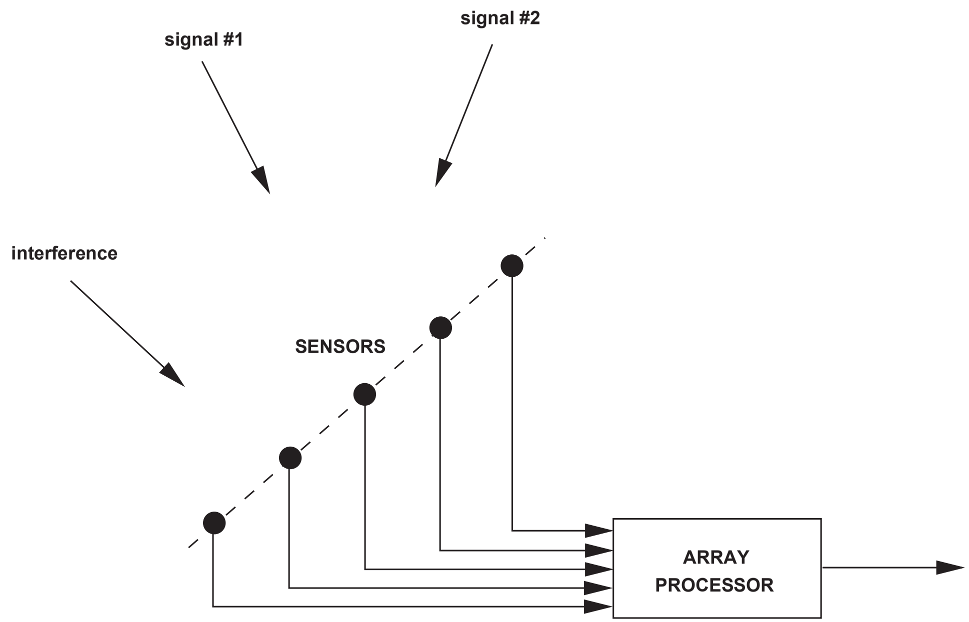
1. Introduction and Fundamentals

General concepts

In this course, signals and noise...

- have spatial dependence
- must be characterized as space-time processes

An array of sensors is represented in the figure below.



1.2 Signals in Space and Time

Defining operators

Let $\nabla_{\mathbf{x}}(\cdot)$ and $\nabla^2_{\mathbf{x}}(\cdot)$ be the gradient and Laplacian operators, i.e.,

$$\begin{aligned}\nabla_{\mathbf{x}}(\cdot) &= \frac{\partial(\cdot)}{\partial x}\overrightarrow{\imath}_x + \frac{\partial(\cdot)}{\partial y}\overrightarrow{\imath}_y + \frac{\partial(\cdot)}{\partial z}\overrightarrow{\imath}_z \\ &= \left[\frac{\partial(\cdot)}{\partial x} \quad \frac{\partial(\cdot)}{\partial y} \quad \frac{\partial(\cdot)}{\partial z}\right]^T\end{aligned}$$

and

$$\nabla^2_{\mathbf{x}}(\cdot) = \frac{\partial^2(\cdot)}{\partial x^2} + \frac{\partial^2(\cdot)}{\partial y^2} + \frac{\partial^2(\cdot)}{\partial z^2}$$

respectively.

Wave equation

From Maxwell's equations,

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

"vector" wave equation

or, for $s(\mathbf{x}, t)$ a general scalar field,

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2}$$

"scalar" wave equation

where: c is the propagation speed, $\vec{E}$ is the electric field intensity, and $\mathbf{x} = [x\ y\ z]^T$ is a position vector.

Note: From this point onwards the terms *wave* and *field* will be used interchangeably.

Monochromatic plane wave

Now assume $s(\mathbf{x}, t)$ has a complex exponential form,

$$s(\mathbf{x}, t) = Ae^{j(\omega t - k_x x - k_y y - k_z z)}$$

where A is a complex constant and k_x, k_y, k_z, and $\omega \geq 0$ are real constants.

Monochromatic plane wave

Substituting the complex exponential form of $s(\mathbf{x}, t)$ into the wave equation, we have

$$k_x^2 s(\mathbf{x}, t) + k_y^2 s(\mathbf{x}, t) + k_z^2 s(\mathbf{x}, t) = \frac{1}{c^2}\omega^2 s(\mathbf{x}, t)$$

or, after canceling $s(\mathbf{x}, t)$,

$$k_x^2 + k_y^2 + k_z^2 = \frac{1}{c^2}\omega^2$$

constraints to be satisfied
by the parameters
of the scalar field

Monochromatic plane wave

From the constraints imposed by the complex exponential form, $s(\mathbf{x},t) = Ae^{j(\omega t - k_x x - k_y y - k_z z)}$ is

- monochromatic
- plane

For example, take the position at the origin of the coordinate space:

$$\mathbf{x} = [0 \;\; 0 \;\; 0]^T$$

$$s(\mathbf{0},t) = Ae^{j\omega t}$$

Monochromatic plane wave

From the constraints imposed by the complex exponential form, $s(\mathbf{x}, t) = Ae^{j(\omega t - k_x x - k_y y - k_z z)}$ is

- monochromatic
- plane

The value of $s(\mathbf{x}, t)$ is the same for all points lying on the plane

$$k_x x + k_y y + k_z z = C$$

where C is a constant.

Monochromatic plane wave

Defining the *wavenumber vector* $\mathbf{k}$ as

$$\mathbf{k} = [k_x \;\; k_y \;\; k_z]^T$$

we can rewrite the equation for the monochromatic plane wave as

$$s(\mathbf{x}, t) = A e^{j(\omega t - \mathbf{k}^T \mathbf{x})}$$

The planes where $s(\mathbf{x}, t)$ is constant are perpendicular to the wavenumber vector $\mathbf{k}$

Monochromatic plane wave

As the plane wave propagates, it advances a distance $\delta\mathbf{x}$ in δt seconds.

Therefore,

$$s(\mathbf{x}, t) = s(\mathbf{x} + \delta\mathbf{x}, t + \delta t)$$

$$\iff Ae^{j(\omega t - \mathbf{k}^T\mathbf{x})} = Ae^{j\left[\omega(t+\delta t) - \mathbf{k}^T(\mathbf{x}+\delta\mathbf{x})\right]}$$

$$\implies \omega\delta t - \mathbf{k}^T\delta\mathbf{x} = 0$$

Monochromatic plane wave

Naturally the plane wave propagates in the direction of the wavenumber vector, i.e.,

$\mathbf{k}$ and $\delta\mathbf{x}$ point in the same direction.

Therefore,

$$\mathbf{k}^T\delta\mathbf{x} = \|\mathbf{k}\|\,\|\delta\mathbf{x}\|$$

$$\Longrightarrow \omega\delta t = \|\mathbf{k}\|\,\|\delta\mathbf{x}\|$$

or, equivalently,

$$\frac{\|\delta\mathbf{x}\|}{\delta t} = \frac{\omega}{\|\mathbf{k}\|}$$

Remember the <u>constraints</u>?

$$\|\mathbf{k}\|^2 = \omega^2/c^2$$

Monochromatic plane wave

After $T = 2\pi/\omega$ seconds, the plane wave has completed one cycle and it appears as it did before, but its *wavefront* has advanced a distance of one *wavelength*, λ.

For $\|\delta \mathbf{x}\| = \lambda$ and $\delta t = T = \frac{2\pi}{\omega}$,

$$T = \frac{\lambda \|\mathbf{k}\|}{\omega} \Longrightarrow \lambda = \frac{2\pi}{\|\mathbf{k}\|}$$

The wavenumber vector, $\mathbf{k}$, may be considered a *spatial frequency* variable, just as ω is a *temporal frequency* variable.

Monochromatic plane wave

We may rewrite the wave equation as

$$
\begin{aligned}
s(\mathbf{x}, t) &= Ae^{j(\omega t - \mathbf{k}^T \mathbf{x})} \\
&= Ae^{j\omega(t - \boldsymbol{\alpha}^T \mathbf{x})}
\end{aligned}
$$

where $\boldsymbol{\alpha} = \mathbf{k}/\omega$ is the *slowness vector*.

As $c = \omega/\|\mathbf{k}\|$, vector $\boldsymbol{\alpha}$ has a magnitude which is the reciprocal of c.

Periodic propagating periodic waves

Any arbitrary periodic waveform $s(\mathbf{x}, t) = s(t - \boldsymbol{\alpha}^T \mathbf{x})$ with fundamental period ω_0 can be represented as a sum:

$$s(\mathbf{x}, t) = s(t - \boldsymbol{\alpha}^T \mathbf{x}) = \sum_{n=-\infty}^{\infty} S_n e^{jn\omega_0 (t - \boldsymbol{\alpha}^T \mathbf{x})}$$

The coefficients are given by

$$S_n = \frac{1}{T} \int_0^T s(u) e^{-jn\omega_0 u} du$$

Periodic propagating periodic waves

Based on the previous derivations, we observe that:

- The various components of $s(\mathbf{x}, t)$ have different frequencies $\omega = n\omega_0$ and different wavenumber vectors, $\mathbf{k}$.
- The waveform propagates in the direction of the slowness vector $\boldsymbol{\alpha} = \mathbf{k}/\omega$.

Nonperiodic propagating waves

More generally, any function constructed as the integral of complex exponentials who also have a defined and converged Fourier transform can represent a waveform

$$s(\mathbf{x}, t) = s(t - \boldsymbol{\alpha}^T \mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega(t - \boldsymbol{\alpha}^T \mathbf{x})} d\omega$$

where

$$S(\omega) = \int_{-\infty}^{\infty} s(u) e^{-j\omega u} du$$

We will come <u>back</u> to this later...