

Capítulo 9

A Série e Integral de Fourier

Sumário

- A Forma Trigonométrica da Série de Fourier
- A Resposta a Entradas Periódicas
- A Forma Complexa da Série de Fourier
- A Transformada de Fourier
- A Resposta a Entradas Não-Periódicas
- Problemas

9.1 A Forma Trigonométrica da Série de Fourier

- Para uma função periódica, $f(t+T)=f(t)$:

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \text{sen}(n\omega_0 t)]$$

- Uma função $f(t)$ é ortogonal a $g(t)$, ambas periódicas, quando:

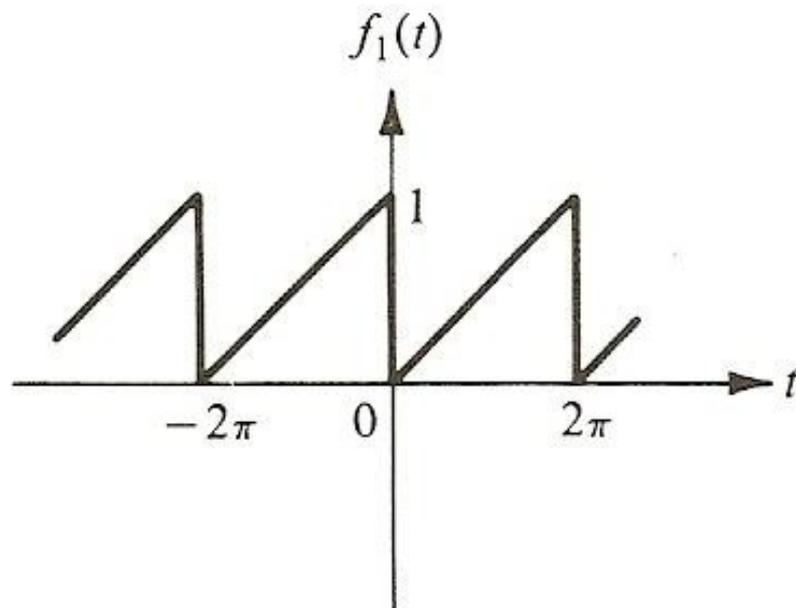
$$\int_t^{t+T} f(t)g(t) dt = 0$$

- Logo: $\int_0^T f(t) \cos n\omega_0 t dt = a_n \int_0^T \cos^2 n\omega_0 t dt \rightarrow a_n = \frac{1}{T} \int_0^T f(t) \cos n\omega_0 t dt$

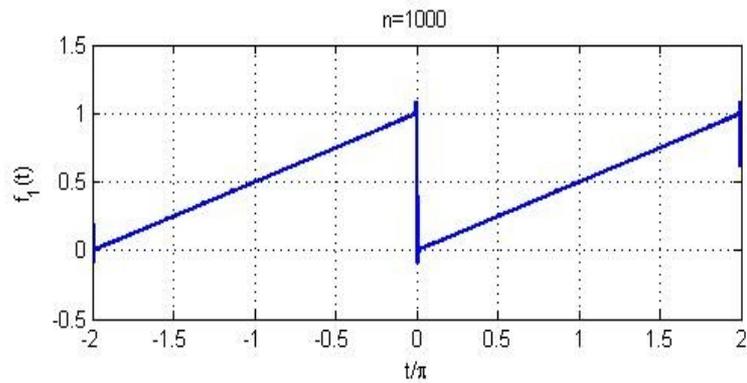
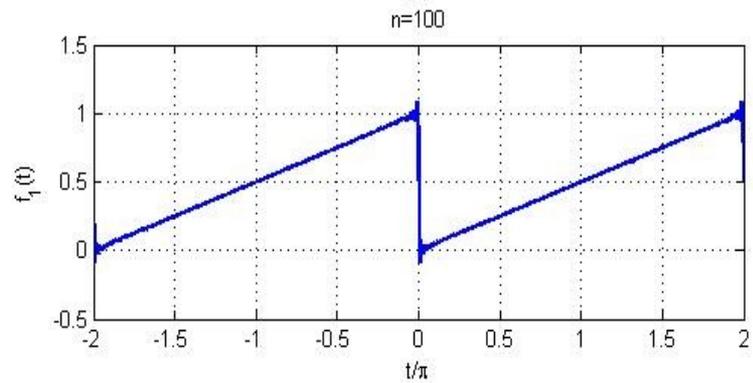
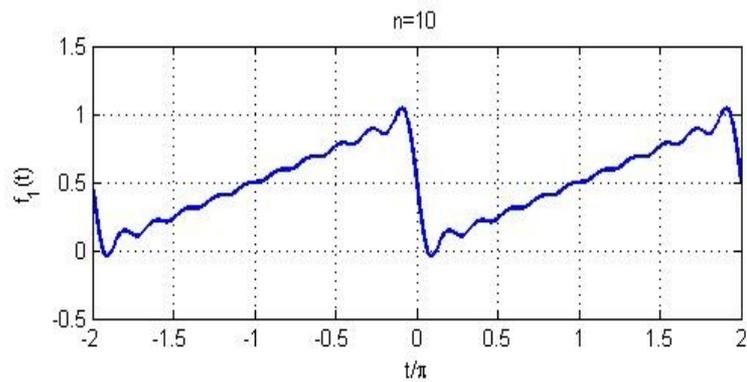
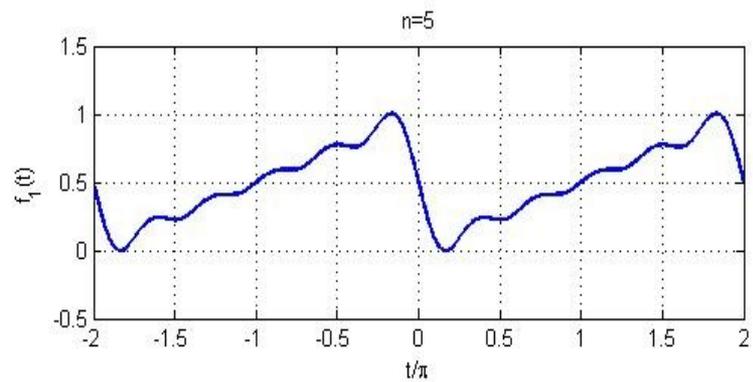
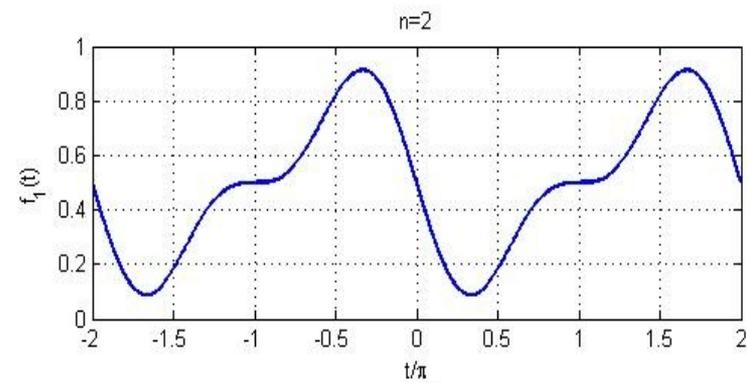
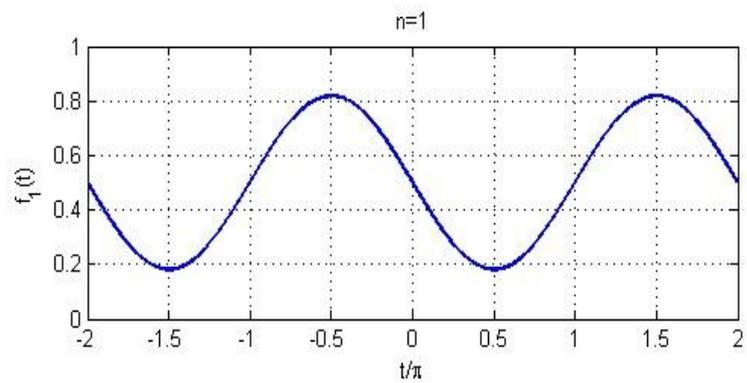
$$\int_0^T f(t) \text{sen} n\omega_0 t dt = b_n \int_0^T \text{sen}^2 n\omega_0 t dt \rightarrow b_n = \frac{1}{T} \int_0^T f(t) \text{sen} n\omega_0 t dt$$

... e $\int_0^T f(t) dt = a_0 \int_0^T dt \rightarrow a_0 = \frac{1}{T} \int_0^T f(t) dt$

- Exemplo:



- a_0 (por inspeção, ver nível DC) vale $1/2$
- $a_n = 0$ (tirando o nível DC, $f_1(t) - 1/2$, a função fica ímpar)
- $b_n = -1/(n\pi)$ OBS: função par $\rightarrow b_n = 0$
- Vejamos o efeitos das harmônicas numa figura em Matlab ...



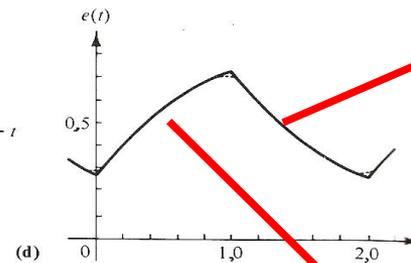
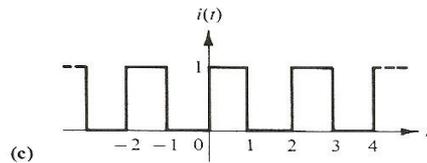
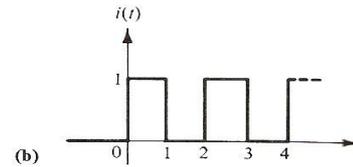
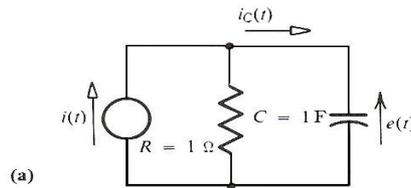
Combinando senos e cossenos, a forma trigonométrica da Série de Fourier pode ser dada como:

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

- Onde $A_n = \sqrt{a_n^2 + b_n^2}$ e $\phi_n = -\text{arc tg} \frac{b_n}{a_n}$
- Observe que para cada frequência, temos uma informação de módulo e uma de fase!

9.2 A Resposta a Entradas Periódicas

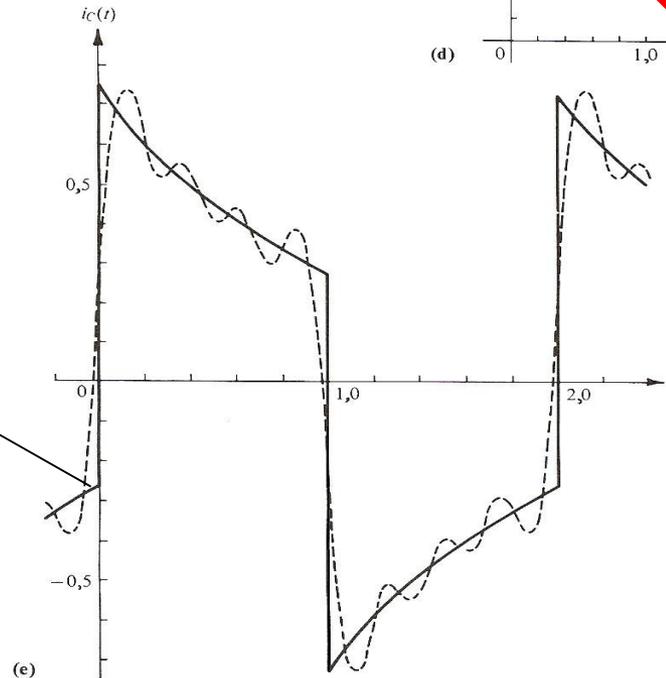
- Dado o circuito e sua entrada $i(t)$, pede-se a tensão de saída $e(t)$.



$$\frac{\epsilon}{1 + \epsilon} \epsilon^{-t}$$

$$1 - \frac{\epsilon}{1 + \epsilon} \epsilon^{-t}$$

4 harmônicas não nulas



- As três etapas básicas nesta análise via Fourier:

- determina-se a série de Fourier da entrada

$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

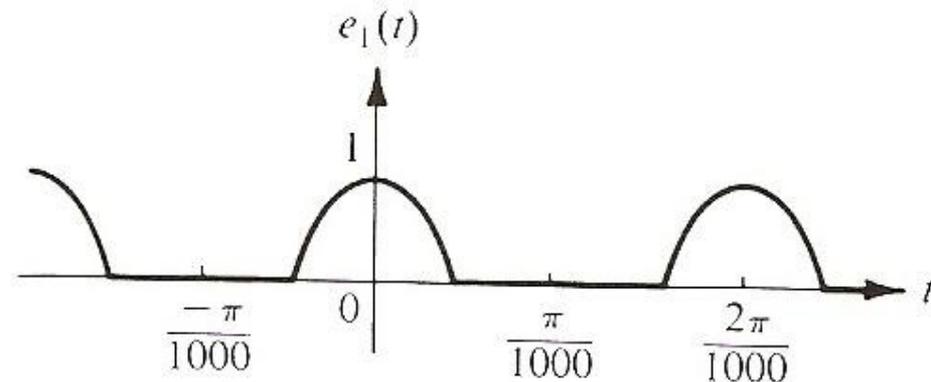
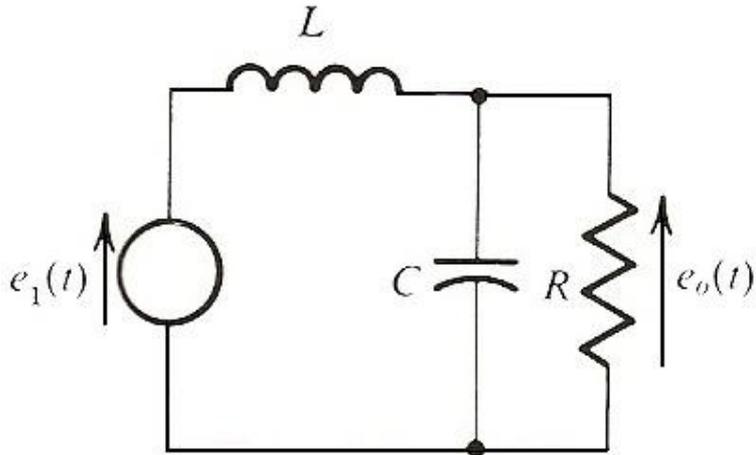
- computa-se a série da saída

$$y(t) = a_0 H(0) + \sum_{n=1}^{\infty} A_n |H(jn\omega_0)| \cos[n\omega_0 t + \phi_n + \angle H(jn\omega_0)]$$

onde $H(j\omega)$ é a função de circuito calculada em ω

- combina-se um número suficiente de harmônicas para ter-se uma forma de onda de saída

- O filtro PB da figura é projetado para produzir uma tensão de saída quase constante para a "senóide retificada em meia-onda". Se $L=1\text{H}$ e $R=1000\Omega$, determine C tal que o maior valor de pico do maior componente AC da saída se $1/20$ do componente DC.



- Determine a resposta (forçada) $e_o(t)$ à entrada $e_1(t)$.

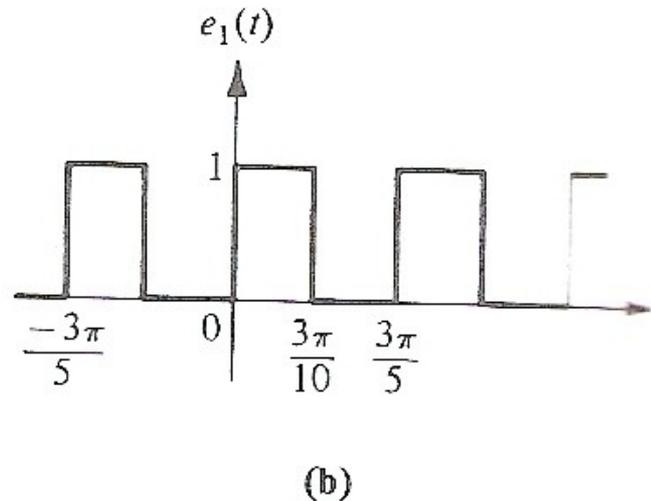
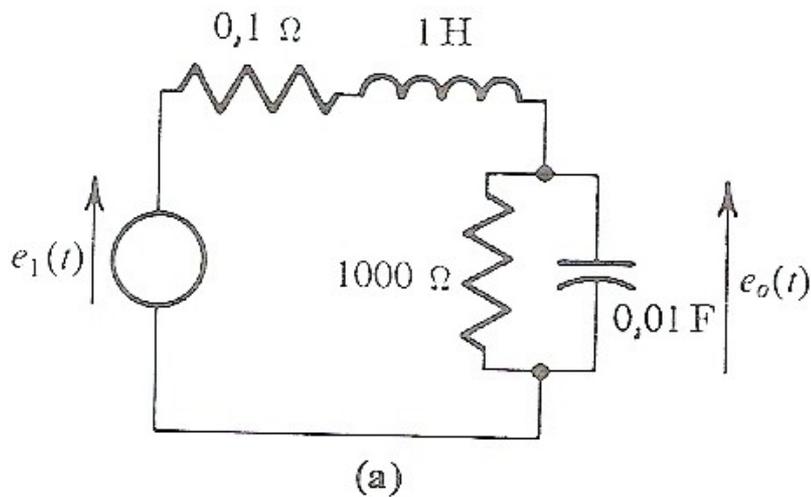
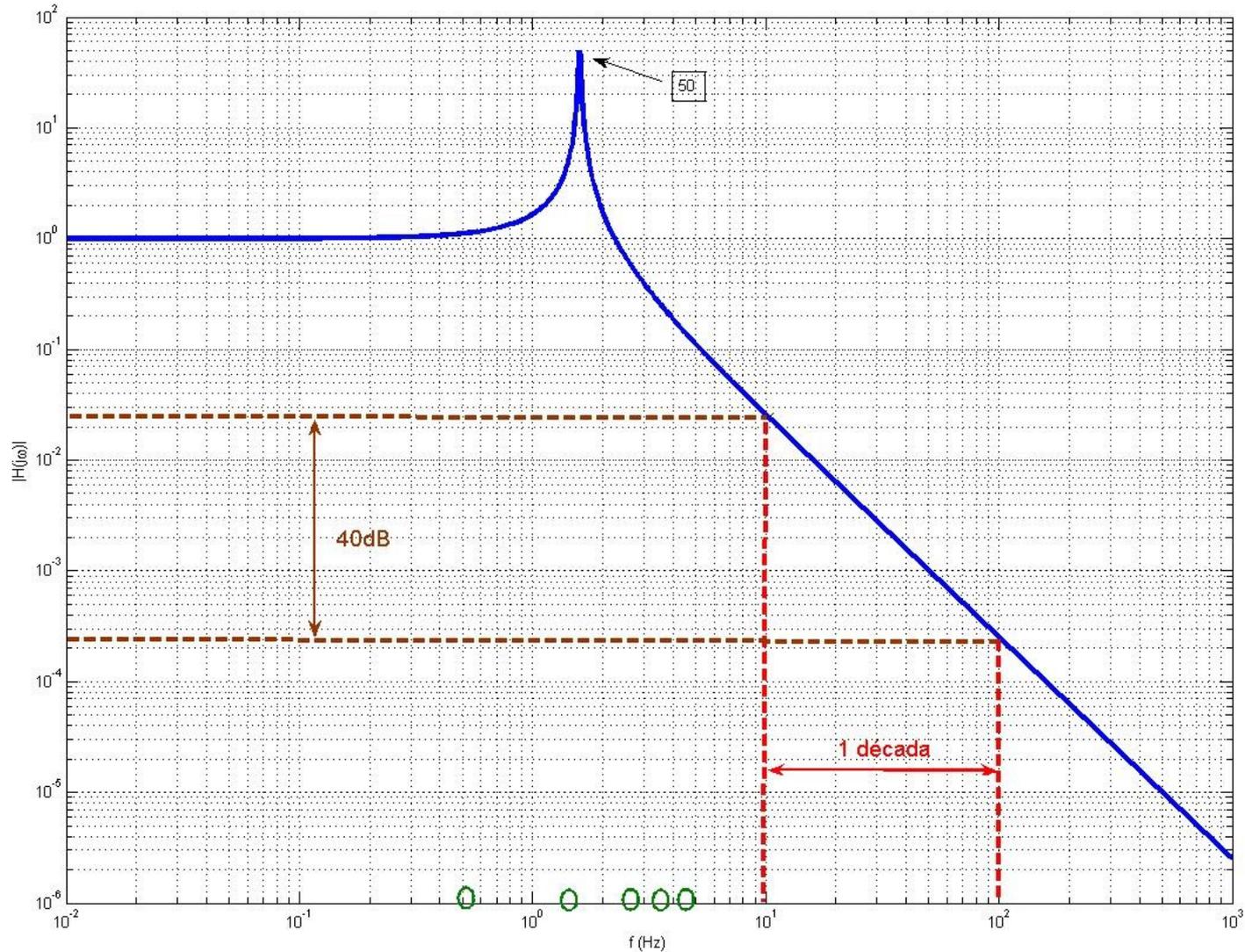


Fig. 9.2-3

- $e_0(t) \approx 50(2/3\pi)\text{sen}(10t-90^\circ)$



- O sinal AM:

$$\begin{aligned}x(t) &= [1 - k\cos(\beta_1 t)]\cos(\beta_2 t), \beta_2 \gg \beta_1 \\ &= \cos\beta_2 t - (k/2)\cos(\beta_1 + \beta_2)t - \\ &\quad (k/2)\cos(\beta_2 - \beta_1)t\end{aligned}$$

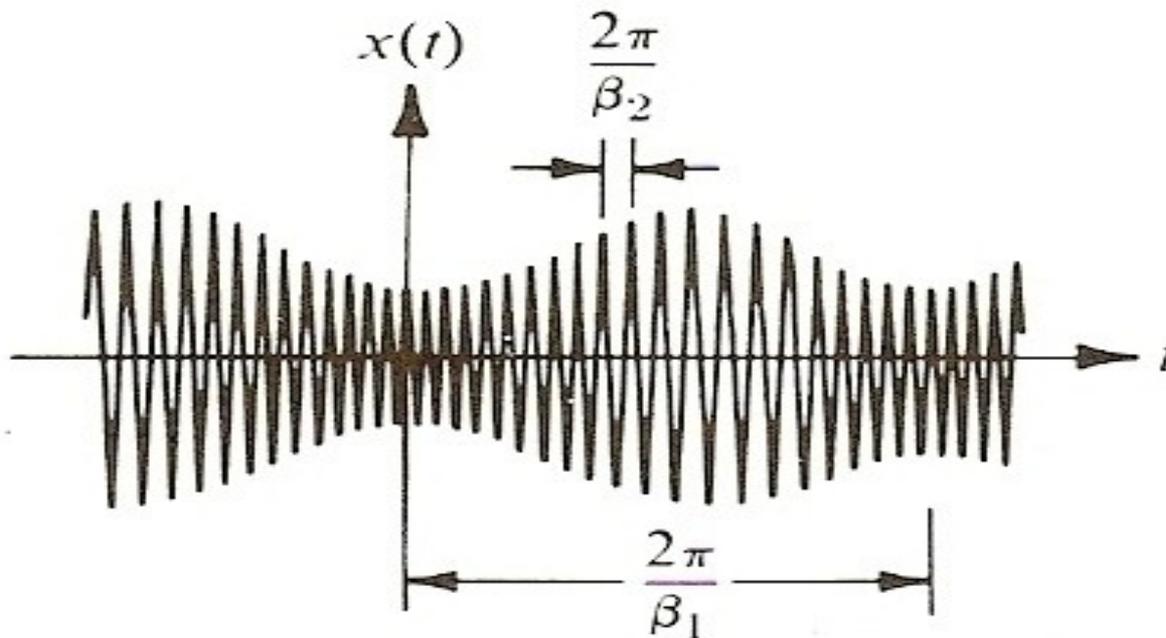


Fig. 9.2-4



9.3 A Forma Complexa da Série de Fourier

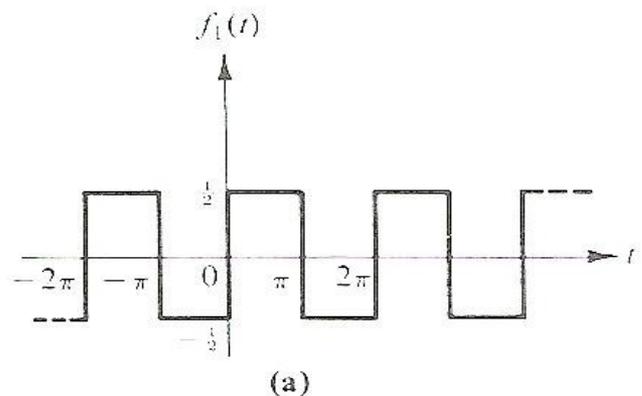
- Expressando senos e cossenos por suas relações envolvendo exponenciais complexas, a Série de Fourier pode ser escrita como:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

onde

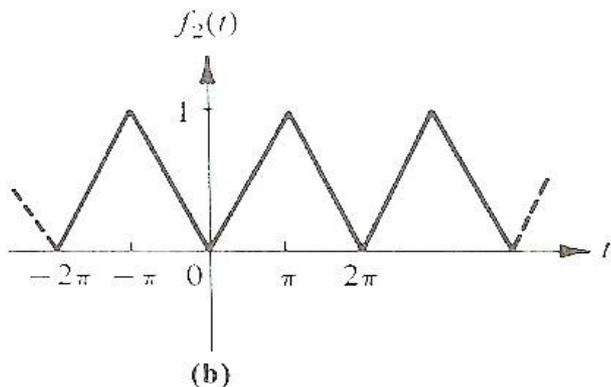
$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-jnt} dt$$

- Coeficientes de algumas formas de onda:



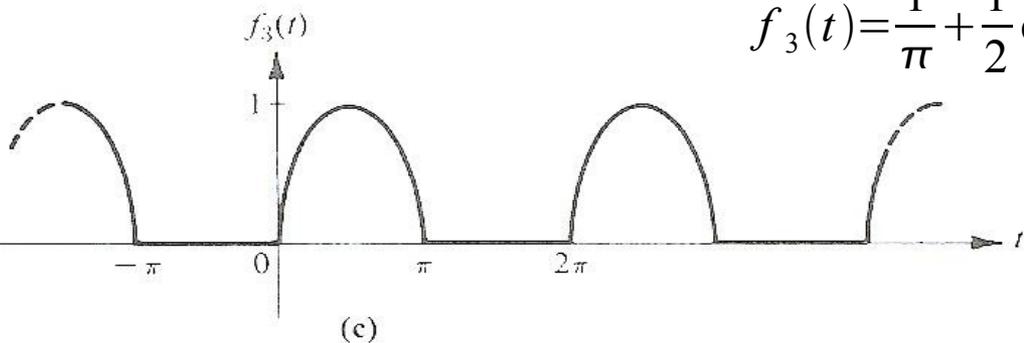
$$f_1(t) = \frac{2}{\pi} \left(\text{sen } t + \frac{1}{3} \text{sen } 3t + \frac{1}{5} \text{sen } 5t + \dots \right)$$

$$c_n = -\frac{j}{n\pi}$$



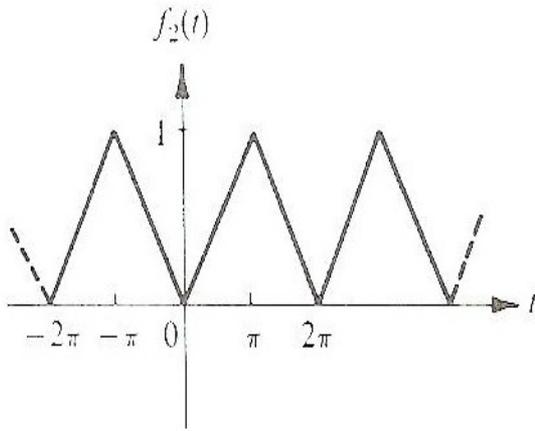
$$f_2(t) = \frac{1}{2} - \frac{4}{\pi^2} \left(\cos t + \frac{1}{9} \cos 3t + \dots \right)$$

$$c_n = \frac{1}{2}, n=0 \text{ e } \frac{-2}{n^2\pi^2}, n \text{ ímpar (0 se } n \text{ par)}$$



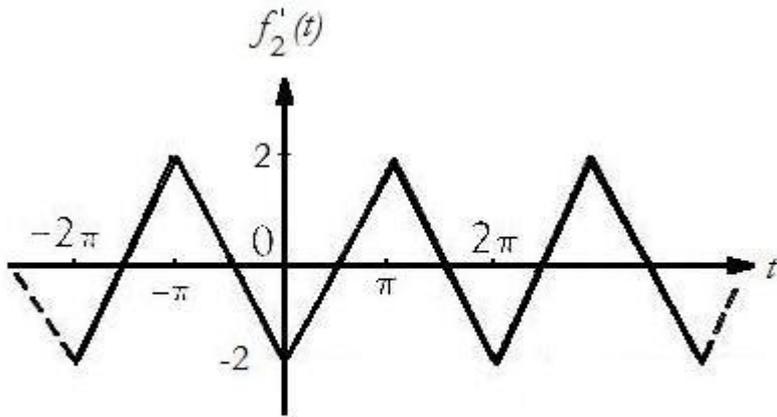
$$f_3(t) = \frac{1}{\pi} + \frac{1}{2} \cos(t - 90^\circ) - \frac{2}{3\pi} \cos 2t - \frac{2}{15\pi} \cos 4t - \dots$$

$$c_n = -\frac{1}{2\pi} \left(\frac{1 + e^{-jn\pi}}{n^2 - 1} \right) \quad 17$$



$$f_2(t) = \frac{1}{2} - \frac{4}{\pi^2} \left(\cos t + \frac{1}{9} \cos 3t + \dots \right)$$

$$c_n = \frac{1}{2}, n=0 \text{ e } \frac{-2}{n^2 \pi^2}, n \text{ ímpar (0 se n par)}$$



$$f'_2(t) = 4f_2(t) - 2$$

$$f''_2(t) = f_2(t - \psi) \rightarrow c''_n = c_n e^{-jn\omega\psi}$$

- **A resposta a sinais periódicos:** se a entrada é um "auto-sinal", a saída é uma constante (auto-valor) multiplicando o mesmo sinal de entrada

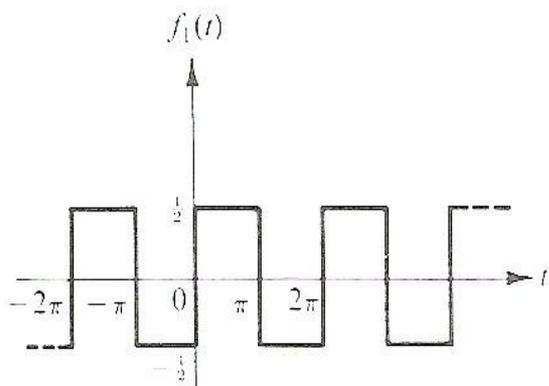
$$e^{j\omega t} \rightarrow H(j\omega)e^{j\omega t}$$

- Logo, $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \rightarrow y(t) = \sum_{n=-\infty}^{\infty} [c_n H(jn\omega_0)] e^{jn\omega_0 t}$

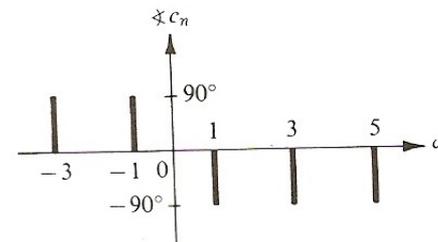
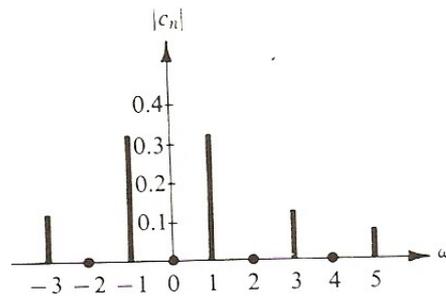
- Portanto $c_n^{(y)} = H(jn\omega_0)c_n^{(x)}$

$$\dots \quad |c_n^{(y)}| = |H(jn\omega_0)| \cdot |c_n^{(x)}| \quad e \quad \angle c_n^{(y)} = \angle H(jn\omega_0) + \angle c_n^{(x)}$$

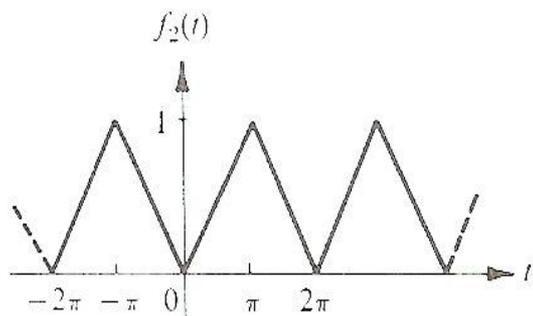
- O espectro de frequência de sinais **periódicos**: a resposta em frequência $H(j\omega)$, em módulo e fase, sugere que c_n em função de ω seria uma maneira útil de representar os harmônicos de um sinal periódico
- Os gráficos de módulo e fase de c_n terão valores somente em frequências angulares discretas e são chamados de espectro discreto ou de espectro de barras
- Eventualmente, representa-se também espectro discreto por $A_n \angle \varphi_n$ ($= 2c_n$)



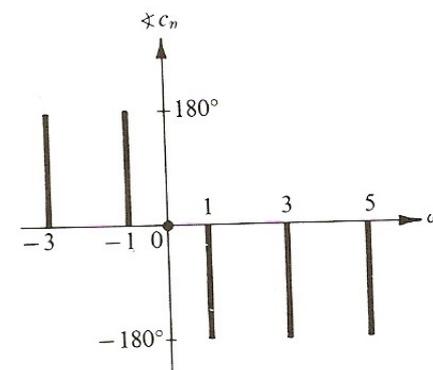
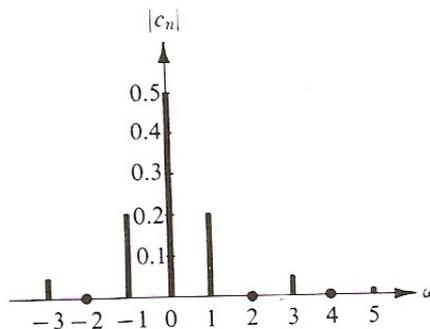
(a)



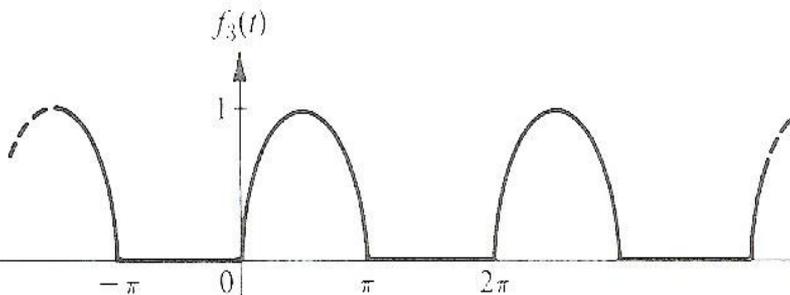
(a) Espectro de frequência para $f_1(t)$



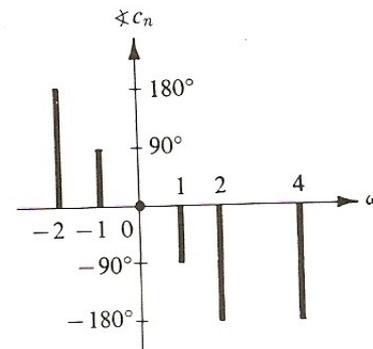
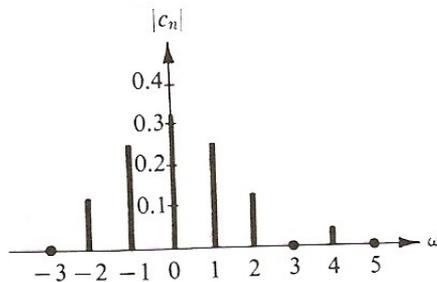
(b)



(b) Espectro de frequência para $f_2(t)$

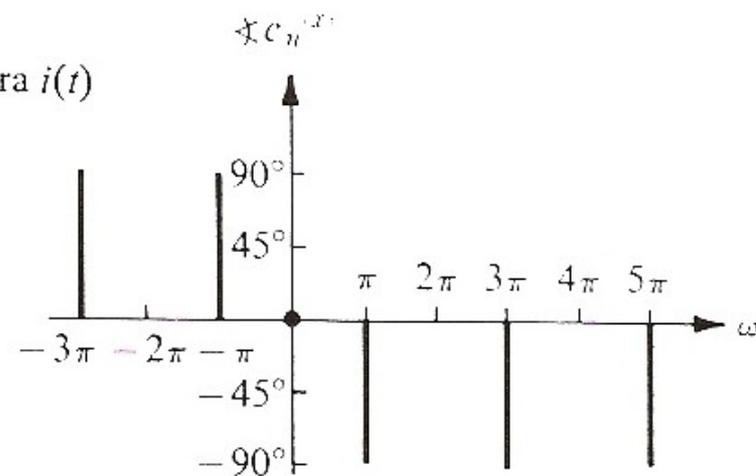
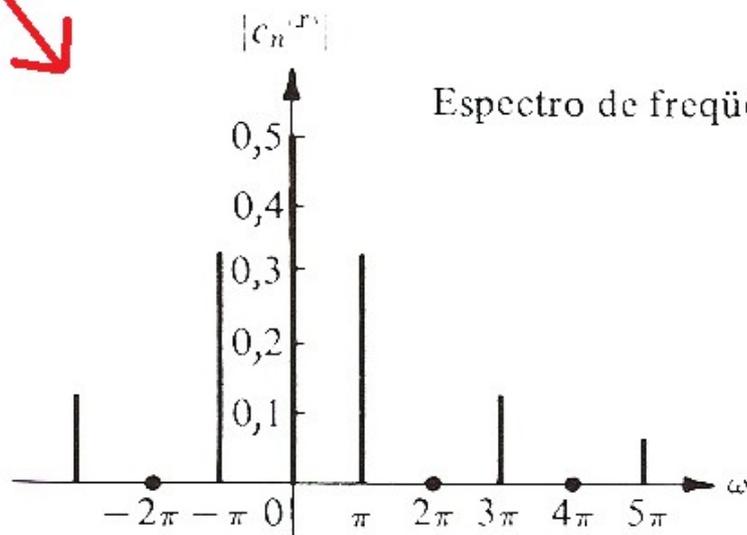
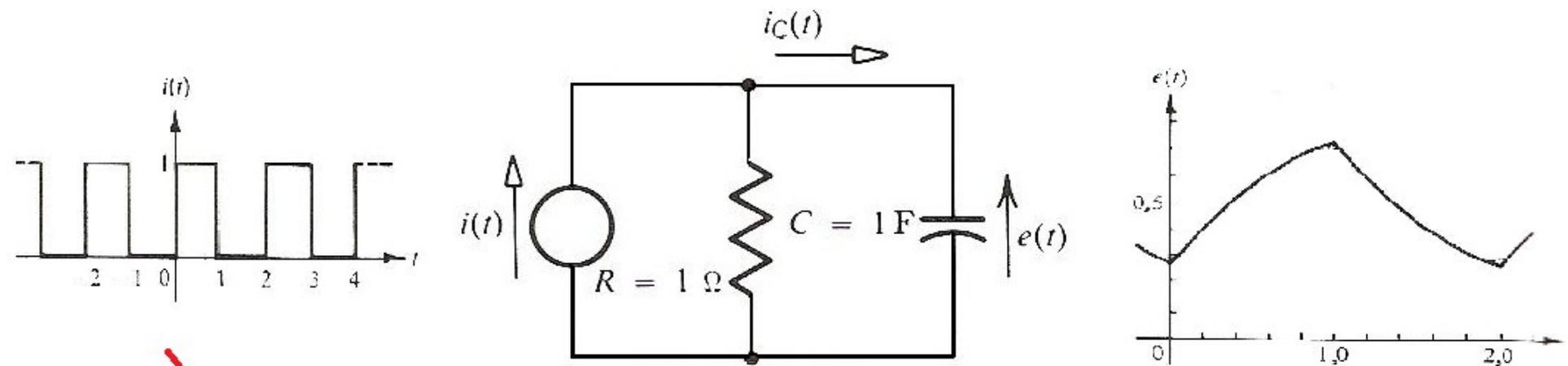


(c)

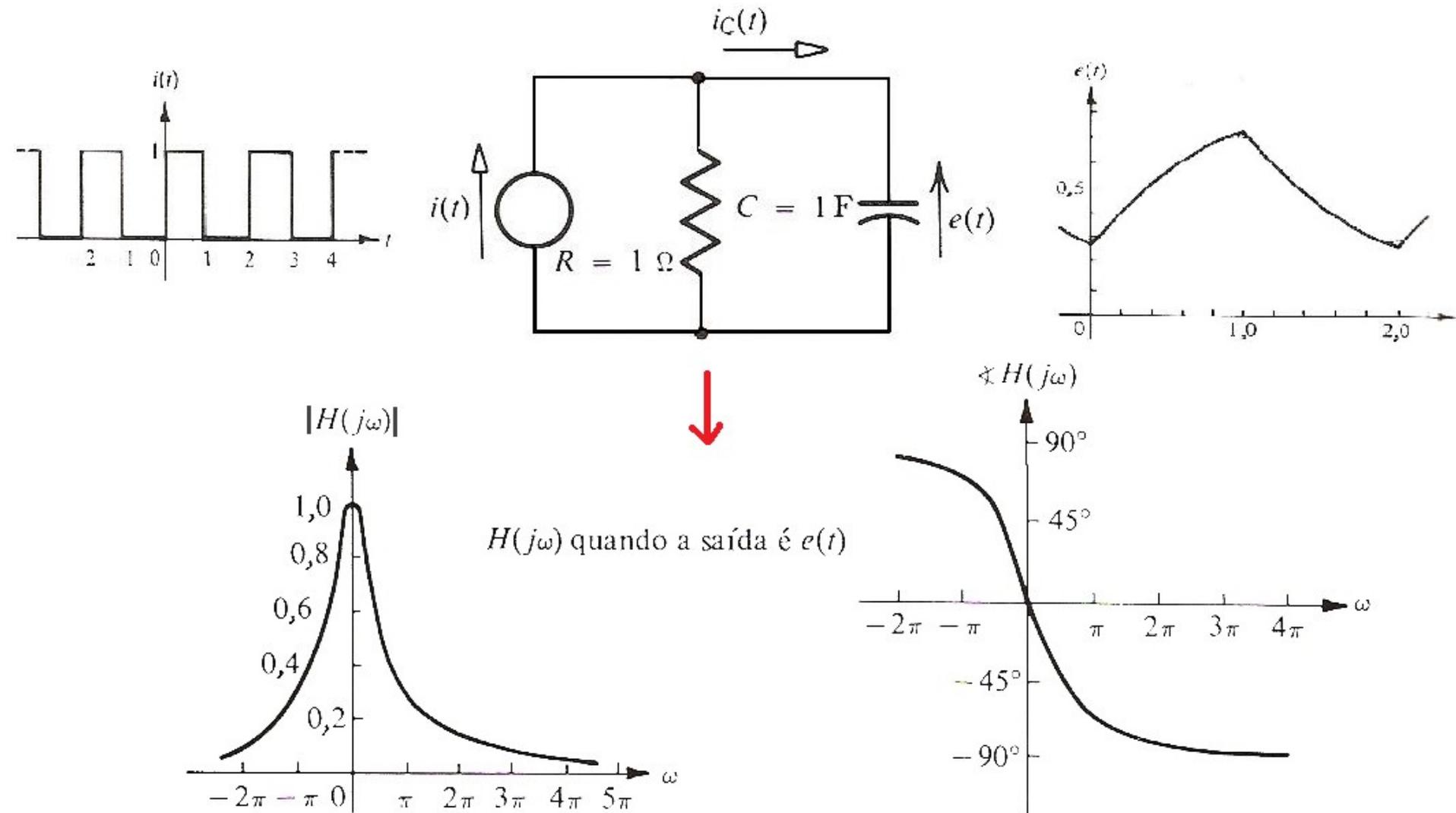


(c) Espectro de frequência para $f_3(t)$

- No circuito abaixo, tomemos o espectro da entrada $i(t)$...

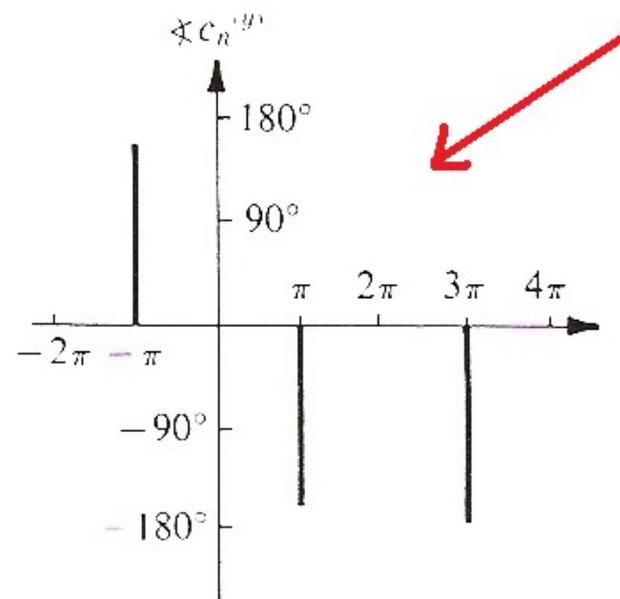
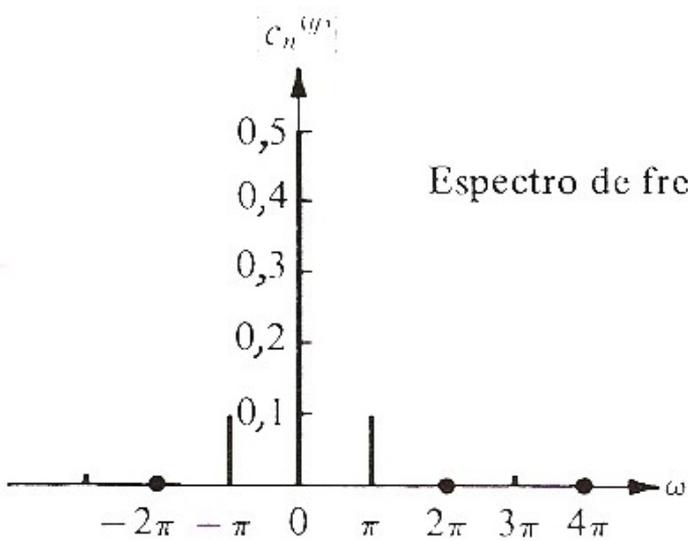
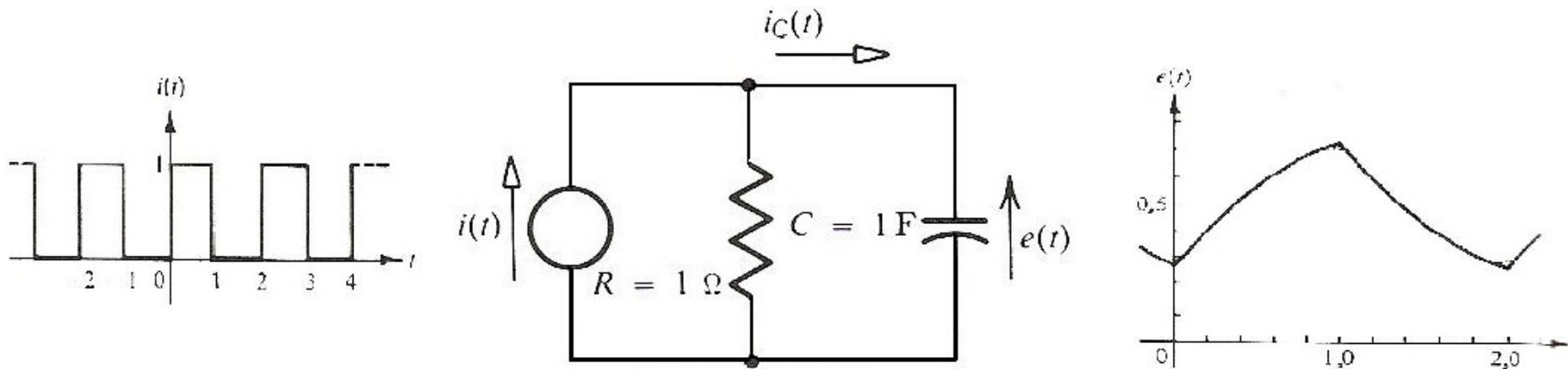


- ... da resposta em frequência do circuito:

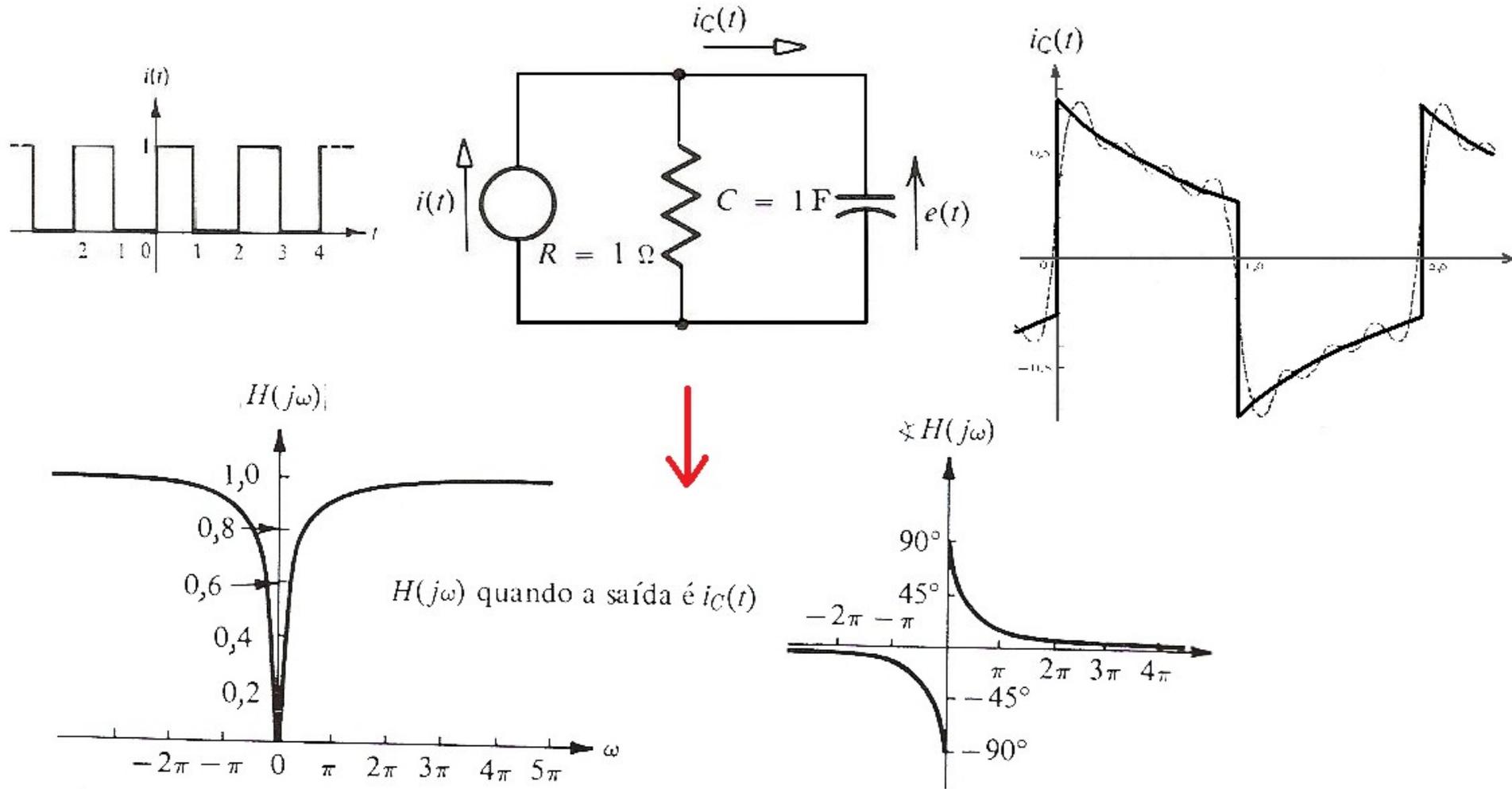


$H(j\omega)$ permite **passar as baixas** frequências!

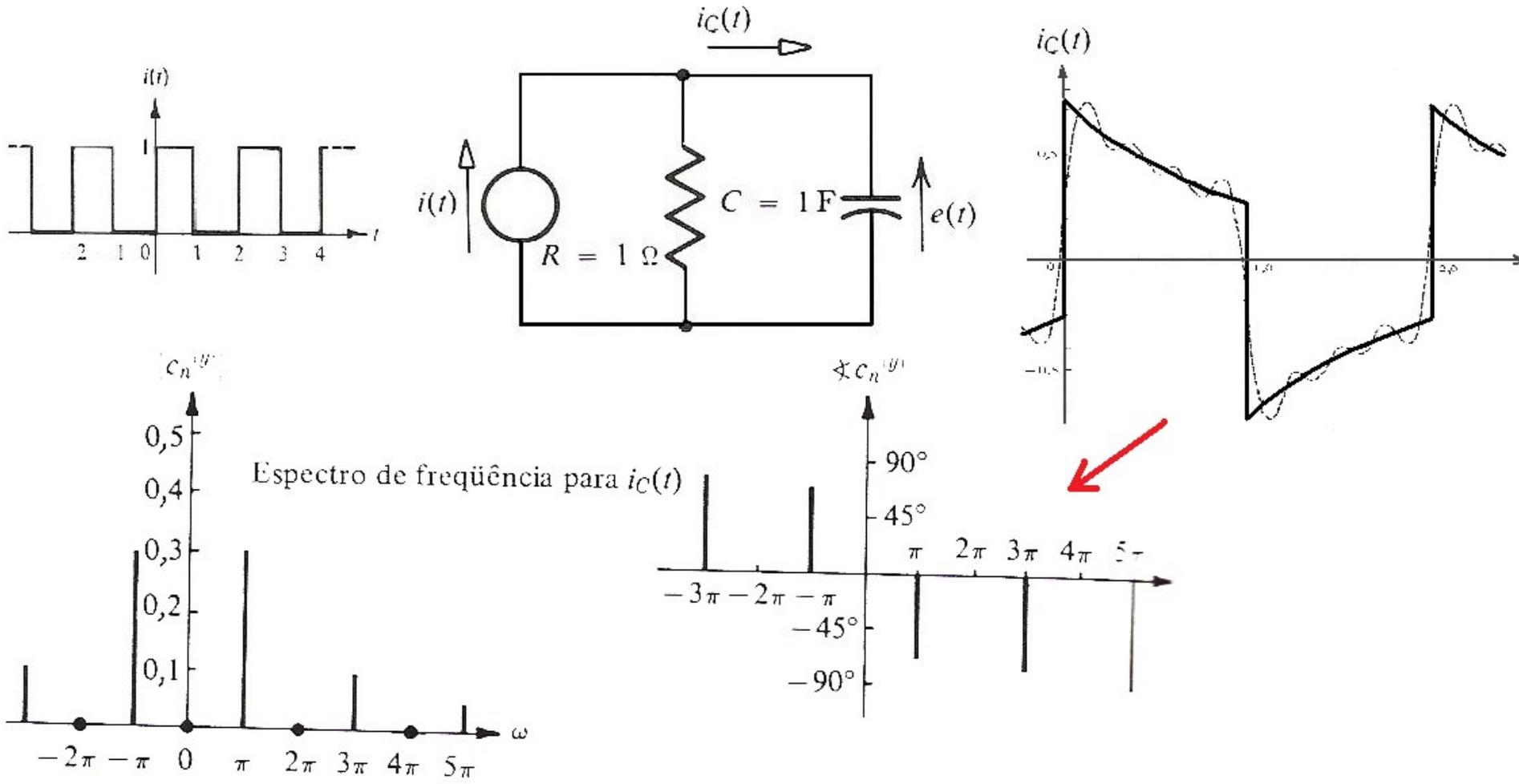
- ... e da saída $e(t)$:



- Se a saída fosse $i_C(t)$, teríamos:



- Se a saída fosse $i_c(t)$, teríamos:



- O efeito do aumento da distância entre pulsos sucessivos: em muitas aplicações, são repetidos pulsos de uma forma de onda fixa

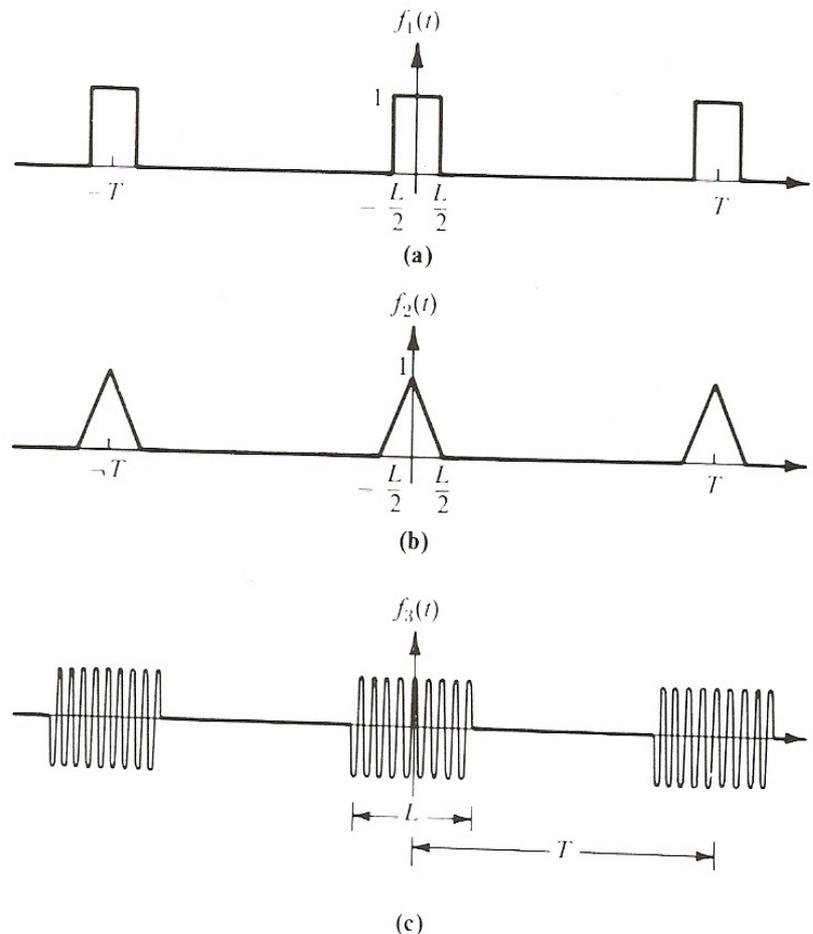
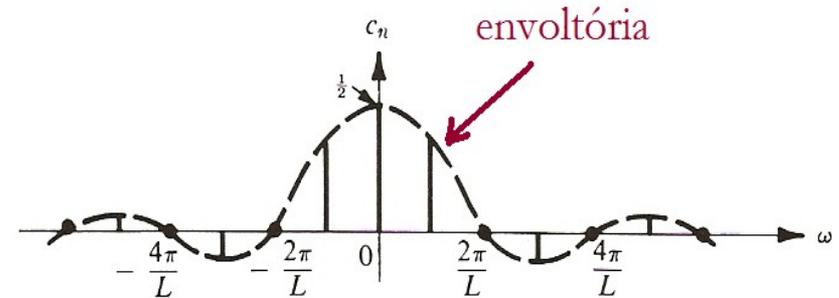
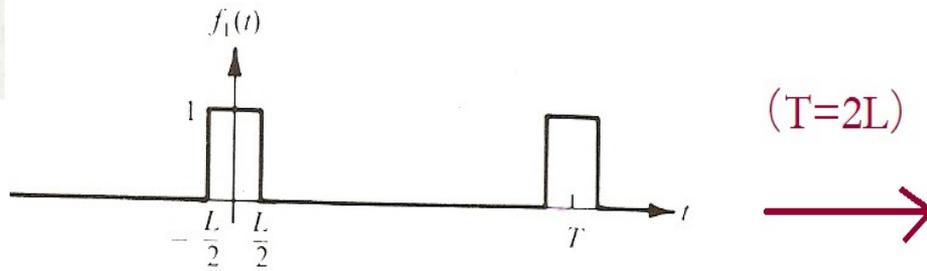
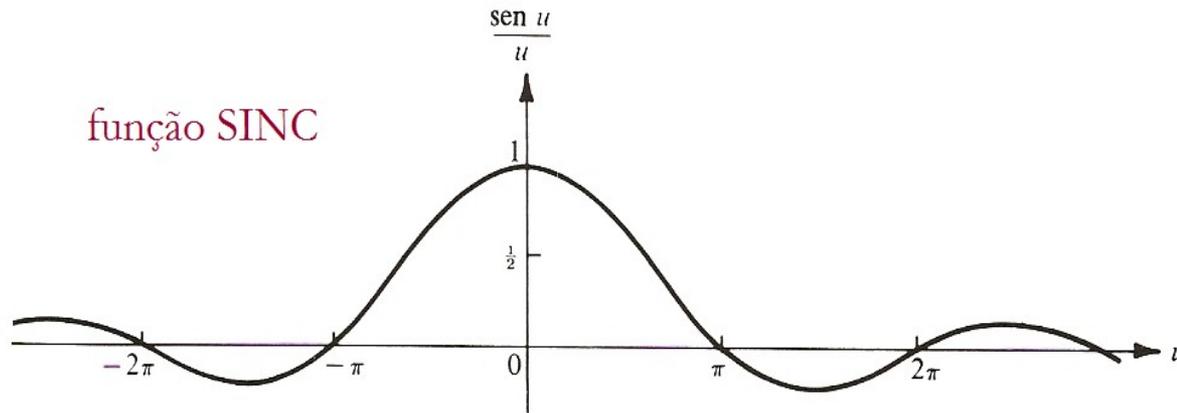
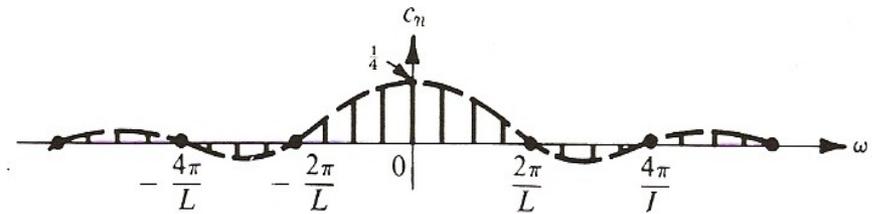


Fig. 9.3-5

- Espectro de $f_1(t)$. Note que $c_n = \frac{L}{T} \left[\frac{\text{sen}(\omega L/2)}{\omega L/2} \right]$



Se fizermos $T=4L$
(duplicamos o período)



$T \rightarrow \infty$: sinal não periódico & espectro contínuo

9.4 A Transformada de Fourier

9.5 A Resposta a Entradas Não-Periódicas

(VER ANOTAÇÕES DE AULA)

Problemas